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Total orderings defined on the set of all fuzzy numbers

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Abstract

In this work, a new concept of upper dense sequence in interval $(0, 1]$ is introduced. There are infinitely many upper dense sequences in interval $(0, 1]$. Using any upper dense sequence, a new decomposition theorem for fuzzy sets is established and proved. Then, using a chosen upper dense sequence as one of the necessary reference systems, infinitely many total orderings on the set of all fuzzy numbers can be well defined. Among them, a common upper dense sequence based on the binary numbers is suggested as a natural default option. Another upper dense sequence based on the rational numbers is also suggested. Regarding real numbers as special fuzzy numbers, all of these total orderings defined by using the suggested upper dense sequences are consistent with the natural ordering of real numbers. Building total ordering on the set of all fuzzy numbers in such a way is significant for fuzzy data analysis and, therefore, may be used in decision making with fuzzy information.
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1. Introduction

Fuzzy numbers is one of the most important mathematical concepts concerning fuzziness. Ranking fuzzy numbers is an essential step in analyzing fuzzy information in optimization, data mining, decision making, and related areas. Since the introduction of the concept of fuzzy sets and fuzzy numbers in the sixties of the last century, many significant contributions have been made in ranking fuzzy numbers [1–4,7–9,14–17,19,20,22]. They have respect intuition based on some geometric characteristics (e.g., area, distance, or centroid), and can be used for various purposes. Mostly, these methods can either order a set of fuzzy numbers that are not equivalent according to some selected characteristic(s), that is, only rank fuzzy numbers but allow different fuzzy numbers to have the same rank, or order some special types of fuzzy numbers, such as the triangular (or, more generally, trapezoidal) fuzzy numbers. In addition, a total ordering for the graded numbers, which are similar to some special type of fuzzy numbers and can be identified by finitely many real-valued parameters, has been discussed [10].

Generally, based on one or more characteristics of fuzzy numbers, an equivalence relation and an opposite but transitive relation on the set of all fuzzy numbers can be defined. Using these two relations, it is easy to define a total ordering on its quotient space of the equivalence relation, but not on the set of all fuzzy numbers themselves. Up to

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now, no one has reported any successful result on total orderings defined on the set of all fuzzy numbers in literature. The difficulty of defining total ordering for all fuzzy numbers is that there is no effective tool to identify an arbitrarily given fuzzy number by only finitely many real-valued parameters. In this work, by establishing a new decomposition theorem for fuzzy sets, we can use an upper dense sequence with the natural ordering of real numbers and a chosen ordering for the end points of rectangles as the necessary reference systems to overcome the above-mentioned difficulty and, then, present a way for defining various total orderings on the set of all fuzzy numbers.

This paper is arranged as follows. After the Introduction, some necessary fundamental knowledge on orderings and fuzzy numbers is reviewed in Section 2. In Section 3, we survey existing results on the total ordering defined on several sets of fuzzy numbers with some common special type. Section 4 is used to introduce and discuss the concept of upper dense sequence in interval $(0, 1]$. A new decomposition theorem for fuzzy sets is established in Section 5. Then, we define total orderings on the set of all fuzzy numbers in Section 6. Several examples showing how the total ordering can be used for ranking or ordering fuzzy numbers are presented in Section 7. Finally, conclusions are given in Section 8.

2. Orderings and fuzzy numbers

Let X be a nonempty set. Any subset of the product set $X \times X$ is called a relation, denoted by R, on X. We write aRb if and only if $(a, b) \in R$. Relation R is reflective if and only if aRa for every $a \in X$. Relation R is symmetric if and only if, for any $a, b \in X$, aRb implies bRa. Relation R is antisymmetric if and only if, for any $a, b \in X$, aRb and bRa imply $a = b$. Relation R is transitive if and only if, for any $a, b, c \in X$, aRb and bRc imply aRc. Relation R is called a partial ordering on X if it is reflective, antisymmetric, and transitive. A partial ordering R on X is called a total ordering if either aRb or bRa for any $a, b \in X$. Two total orderings R_1 and R_2 are different if and only if there exist $a, b \in X$ with $a \neq b$ such that aR_1b but bR_2a. For any given total ordered infinite set, there are infinitely many different ways to redefine a new total ordering on it. Relation R is called an equivalent relation if it is reflective, symmetric, and transitive.

Now, let $\mathbb{R} = (-\infty, \infty)$. A fuzzy subset of $\mathbb{R}$, denoted by $\tilde{e}$, is called a fuzzy number if its membership function $m_e : \mathbb{R} \to [0, 1]$ satisfies the following conditions.

(FN1) Set $\{x \mid m_e(x) \geqslant \alpha\}$, the α-cut of $\tilde{e}$ (denoted by e_α), is a closed interval for every $\alpha \in (0, 1]$.
(FN2) Set $\{x \mid m_e(x) > 0\}$, the support set of $\tilde{e}$ (denoted by $\mathrm{supp}(e)$), is bounded.

Condition (FN1) is equivalent to the following three conditions.

(FN1.1) There exists at least one real number a_0 such that $m_e(a_0) = 1$.
(FN1.2) Function m_e is nondecreasing on $(-\infty, a_0]$ and nonincreasing on $[a_0, \infty)$.
(FN1.3) Function m_e is upper semi-continuous, or say, m_e is right-continuous (i.e., $\lim_{x \to x_0+} m_e(x) = m_e(x_0)$) when $x_0 < a_0$ and is left-continuous (i.e., $\lim_{x \to x_0-} m_e(x) = m_e(x_0)$) when $x_0 > a_0$.

For any fuzzy number $\tilde{e}$, set $\{x \mid m_e(x) = 1\}$ is nonempty and is called its core (or, kernel). It is the most important factor for identifying a fuzzy number. Any real number is a special fuzzy number that can be identified by only its core, which is a singleton.

The set of all fuzzy numbers is denoted by $\mathcal{N}_F$. It is easy to define a partial ordering on set $\mathcal{N}_F$ as follows.

First, a partial ordering, denoted by $\preccurlyeq$, on the set of all closed intervals, denoted by $\mathcal{N}_I$, is defined by saying $[a, b] \preccurlyeq [c, d]$ if and only if $a \leqslant c$ and $b \leqslant d$. Then a partial ordering, still denoted by $\preccurlyeq$, on $\mathcal{N}_F$ can be well defined by saying $\tilde{e_1} \preccurlyeq \tilde{e_2}$ if and only if $(e_1)_\alpha \preccurlyeq (e_2)_\alpha$ for all $\alpha \in (0, 1]$.

The major contribution of this work is presented in Section 6, where we give total orderings defined on the set of all fuzzy numbers, $\mathcal{N}_F$.

3. Total orderings defined on some sets of fuzzy numbers with special type

Before defining total orderings on the set of all fuzzy numbers, let us recall some total orderings defined on several common proper subsets of $\mathcal{N}_F$.

Each rectangular fuzzy number can be regarded as a closed interval. So, we can still use $\mathcal{N}_I$, the notation for the set of all interval numbers, to denote the set of all rectangular fuzzy numbers. $\mathcal{N}_I$ is a proper subset of $\mathcal{N}_F$. The membership function of a rectangular number $\tilde{e}$ takes the form

$$m_e(x) = \begin{cases} 1 & \text{if } x \in [a_l, a_r] \\ 0 & \text{otherwise,} \end{cases}$$

and can be simply denoted by vector $[a_l\ a_r]$, where real-valued parameters a_l and a_r satisfy $a_l \leqslant a_r$. From Section 2, we can see that it is easy to define a total ordering on $\mathcal{N}_I$ based on any two selected parameter that can identify a rectangular fuzzy number. For example, one way to define a total ordering $\preccurlyeq$ on $\mathcal{N}_I$ is by a statement that $[a_l\ a_r] \preccurlyeq [b_l\ b_r]$ if and only if $a_l < b_l$ or $a_r \leqslant b_r$ when $a_l = b_l$; another way is by a statement that $[a_l\ a_r] \preccurlyeq [b_l\ b_r]$ if and only if $a_l + a_r < b_l + b_r$ or $a_r - a_l \leqslant b_r - b_l$ when $a_l + a_r = b_l + b_r$. There are infinitely many ways to define different total orderings on $\mathcal{N}_I$. So, there are infinitely many different total orderings on $\mathcal{N}_I$. Each particular total ordering corresponds to (or say, depends on) particularly selected reference systems. For instance, in the above-mentioned examples, the first total ordering corresponding to two selected reference systems: one consists of the set of all real numbers with the natural ordering and another is "left-first, right-second" for end points of intervals (denoted by (l, r)) such an ordered pair. It is easy to see that we can obtain another total ordering, which is different from the above-mentioned two total orderings, on $\mathcal{N}_I$ when "left-first, right-second" is replaced with "right-first, left-second" (denoted by (r, l)) in the second reference system. That is to say, even if two identifying parameters for rectangular fuzzy numbers have been fixed in the second reference system, the defined total ordering on $\mathcal{N}_I$ still depends on the order of these two parameters. People may also use the convex combination of any chosen weight $w \in (0, 1)$ with $w \neq 1/2$ followed by its dual (the convex combination of weight $1 - w$) as the second reference system to define a total ordering on $\mathcal{N}_I$. This is similar to the weighted average of pessimism and optimism expressed in Hurwicz criterion [5,12], where only one coefficient α is used and a ranking on the set of all closed intervals can be obtained. But now, to get a total ordering, we need two ordered indexes as a reference system, i.e., using an ordered pair of w and $1 - w$ satisfying $w \neq 1/2$. In such a way, we have infinitely many different choices for weight w and, therefore, infinitely many different total orderings can be defined on $\mathcal{N}_I$. They depend on the selected reference systems.

Another common special type of fuzzy numbers is the triangular fuzzy numbers whose membership function has a form

$$m_e(x) = \begin{cases} \frac{x-a_l}{a_0-a_l} & \text{if } x \in [a_l, a_0) \\ 1 & \text{if } x = a_0 \\ \frac{x-a_r}{a_0-a_r} & \text{if } x \in (a_0, a_r] \\ 0 & \text{otherwise,} \end{cases}$$

where real-valued parameters a_l, a_0, and a_r satisfy $a_l \leqslant a_0 \leqslant a_r$. The set of all triangular fuzzy numbers is denoted by $\mathcal{N}_T$, which is also a proper subset of $\mathcal{N}_F$. Such a fuzzy number can be simply denoted by vector $[a_l\ a_0\ a_r]$. Thus, a total ordering $\preccurlyeq$ on $\mathcal{N}_T$ may be defined according to the following criterion: $[a_l\ a_0\ a_r] \preccurlyeq [b_l\ b_0\ b_r]$ if and only if

(1) $a_l < b_l$, or
(2) $a_l = b_l$ but $a_0 < b_0$, or
(3) $a_l = b_l$, $a_0 = b_0$, but $a_r \leqslant b_r$.

The above way for defining a total ordering is often referred to as lexicographic in literature [8]. Of course, there are infinitely many different ways, even according to the lexicography, to define a total ordering on $\mathcal{N}_T$.

As a generalization of both rectangular fuzzy numbers and triangular fuzzy numbers, trapezoidal fuzzy numbers are also an important common type of fuzzy numbers, whose membership function has a form

$$m_e(x) = \begin{cases} \frac{x-a_l}{a_b-a_l} & \text{if } x \in [a_l, a_b) \\ 1 & \text{if } x \in [a_b, a_c] \\ \frac{x-a_r}{a_c-a_r} & \text{if } x \in (a_c, a_r] \\ 0 & \text{otherwise,} \end{cases}$$

where a_l, a_b, a_c, and a_r are real-valued parameters with $a_l \leqslant a_b \leqslant a_c \leqslant a_r$. Such a trapezoidal fuzzy number can be simply denoted by vector $[a_l\ a_b\ a_c\ a_r]$. Similar to above discussion for rectangular fuzzy numbers and triangular fuzzy numbers, the lexicography can also be used to define total orderings on the set of all trapezoidal fuzzy numbers. This set is still a proper subset of $\mathscr{N}_F$.

All above-mentioned subsets of $\mathscr{N}_F$ have a common specialty: they involve only finitely many real-valued parameters. Hence, it is easy to define a total ordering on each of them by the lexicography. However, fuzzy numbers in $\mathscr{N}_F$ cannot be identified by only finitely many real-valued parameters. This is a fundamental challenge in defining a total ordering on $\mathscr{N}_F$. In the next two sections, we propose to identify fuzzy numbers in $\mathscr{N}_F$ by infinitely but countably many real-valued parameters such that the lexicography can be used to define total orderings on the set of all fuzzy numbers as well. This entails a new decomposition theorem for fuzzy sets.

4. Upper dense sequences in interval (0, 1]

To define a total ordering on $\mathscr{N}_F$, we need an upper dense sequence, as one of the reference systems, consisting of real numbers in interval $(0, 1]$ described below.

Let D be a set of real numbers in $(0, 1]$, that is, $D \subseteq (0, 1]$.

Definition 1. Set D is upper dense in $(0, 1]$ if, for every point $x \in (0, 1]$ and any $\varepsilon > 0$, there exists $\delta \in D$ such that $\delta \in [x,\ x + \varepsilon)$. Set D is lower dense in $(0, 1]$ if, for every point $x \in (0, 1]$ and any $\varepsilon > 0$, there exists $\delta \in D$ such that $\delta \in (x - \varepsilon, x]$.

Definition 2. Set D is dense in $(0, 1]$ if, for every point $x \in (0, 1]$ and any $\varepsilon > 0$, there exists $\delta \in D$ such that $|x - \delta| < \varepsilon$.

It is evident that D is dense in $(0, 1]$ if D is upper dense or lower dense in $(0, 1]$. The converse statement may be wrong. In fact, a dense set in $(0, 1]$ may not be upper dense in $(0, 1]$ since D being upper dense in $(0, 1]$ implies $1 \in D$ but 1 may not be in D even if D is dense in $(0, 1]$. However, we have the following theorem.

Theorem 1. *If D is dense in $(0, 1]$ and $1 \in D$, then it is upper dense in $(0, 1]$.*

Proof. Assume that D is dense in $(0, 1]$. We only need to show that D is upper dense in $(0, 1)$ due to the fact that $1 \in D$. For any given $x \in (0, 1)$ and $\varepsilon > 0$, take $y \in (x, \min(x + \varepsilon/2, 1)]$. Since D is dense in $(0, 1]$, for $\varepsilon' = \min(y - x, \varepsilon/2) > 0$, there exists $\delta \in D$ such that $|y - \delta| < \varepsilon' \leqslant \varepsilon/2$. Thus, $\delta \in [x, x + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}) = [x, x + \varepsilon)$, i.e., D is upper dense in $(0, 1)$. □

Theorem 2. *If D is dense in $(0, 1]$, then it is lower dense in $(0, 1]$.*

Proof. For any given positive integer n, since D is upper dense in $(0, 1)$, we can find a real number $y_n \in (1 - 1/n, 1) \cap D$. Since $\lim_{n\to\infty} y_n = 1$ and D is dense in $(0, 1)$, we only need to show that D is also lower dense in $(0, 1)$. The proof of this remaining part is similar to that of Theorem 1 and, therefore, is omitted here. □

From Theorems 1 and 2 we know that, in $(0, 1]$, any upper dense set is just a dense set containing real number 1 and is also lower dense.

Any infinite but countable set of real numbers can be expressed as an infinite sequence. Beyond the existence of assigned ordering for their members, the difference between a countable set and a sequence is that the former contains only distinct members but the latter allows repeated occurrences. For example, the collection of all positive integers $\{1, 2, 3, \dots\}$ is a set but $(0, 1, 0, 2, 0, 3, \dots)$, which can also be expressed as $\{a_i \mid i = 1, 2, \dots\}$ with $a_i = (i + (-1)^i i)/4$, is a sequence. Anyway, we may use the notations of belonging and set inclusion in set theory for sequences without any confusion. For example, if S is a sequence consisting of points in set A, we may write $S \subseteq A$; if x is a term of sequence S, we may write $x \in S$. In Sections 5 and 6, we use a given upper dense sequence consisting of infinitely but countably many numbers in interval $(0, 1]$ to establish a new decomposition theorem for fuzzy sets and, then, define total orderings on the set of all fuzzy numbers.

Two examples of upper dense sequences in $(0, 1]$ are given as follows.

Example 1. Let $S_b = \{d_{bi} \mid i = 1, 2, \ldots\}$ be the sequence of all binary numbers in $(0, 1]$ with finitely many bits, where $d_{b1} = 1$, $d_{b2} = \frac{1}{2} = 0.5$, $d_{b3} = \frac{1}{4} = 0.25$, $d_{b4} = \frac{3}{4} = 0.75$, $d_{b5} = \frac{1}{8} = 0.125$, $d_{b6} = \frac{3}{8} = 0.375$, $d_{b7} = \frac{5}{8} = 0.625$, $d_{b8} = \frac{7}{8} = 0.875$, $d_{b9} = \frac{1}{16} = 0.0625, \ldots$, decimally. Generally, we may express

$$d_{bi} = \left[2\left(i - 2^{\lceil \log_2 i \rceil - 1}\right) - 1\right] / 2^{\lceil \log_2 i \rceil}, \quad i = 1, 2, \ldots,$$

where notation $\lceil t \rceil$ denotes the value of the ceiling function at real number t, i.e., the smallest integer not smaller than t [18]. Sequence S_b is upper dense in $(0, 1]$. In the next two sections, it is used as the standard upper dense sequences in $(0, 1]$ for defining total orderings on $\mathscr{N}_F$.

Example 2. Let $S_r = \{d_{ri} \mid i = 1, 2, \ldots\}$, the set of all rational numbers in $(0, 1]$, where $d_{r1} = 1$, $d_{r2} = 1/2$, $d_{r3} = 1/3$, $d_{r4} = 2/3$, $d_{r5} = 1/4$, $d_{r6} = 3/4$, $d_{r7} = 1/5$, $d_{r8} = 2/5$, $d_{r9} = 3/5, \ldots$. Sequence S_r is upper dense in $(0, 1]$ too. If we allow a number to have multiple occurrences in the sequence, the general members in upper dense sequence $S'_r = \{d'_{ri} \mid i = 1, 2, \ldots\}$ can be expressed by

$$d'_{ri} = \frac{i}{k} - \frac{k-1}{2}, \quad i = 1, 2, \ldots,$$

where

$$k = \left\lceil \sqrt{2i + \frac{1}{4}} - \frac{1}{2} \right\rceil.$$

That is, $d'_{r1} = 1$, $d'_{r2} = 1/2$, $d'_{r3} = 2/2$, $d'_{r4} = 1/3$, $d'_{r5} = 2/3$, $d'_{r6} = 3/3$, $d'_{r7} = 1/4$, $d'_{r8} = 2/4$, $d'_{r9} = 3/4, \ldots$. In sequence S'_r, for instance, d'_{r3} is the same real number as d'_{r1}. Though sequences S_r and S'_r are different, we can see in Section 6 that the two total orderings defined on $\mathscr{N}_F$ by using them are the same.

Obviously, for a given upper dense sequence in $(0, 1]$, there are infinitely many different rearrangements that are still upper dense in $(0, 1]$. For example, $(1, 1/2, 3/4, 1/4, 7/8, 5/8, \ldots)$ is a rearrangement of S_b in Example 1. Each upper dense sequence in $(0, 1]$ can be used to define a total ordering on $\mathscr{N}_F$.

Even though a lower dense sequence is necessary for establishing a new decomposition theorem in the next section, we still prefer to use an upper dense sequence since number 1 plays a very important role for identifying fuzzy numbers and number 1 may not be included in a lower dense sequence in interval $(0, 1]$.

5. A new decomposition theorem for fuzzy sets

Before establishing a new decomposition theorem, we recall existing decomposition theorems [6,13,21,23] for fuzzy sets.

Let X be the nonempty universal set and A be a fuzzy subset of X with membership function m_A. The α-cut and the strong α-cut of A are denoted by A_α and $A_{\alpha+}$ respectively, that is, $A_\alpha = \{x \mid m_A(x) \geqslant \alpha,\ x \in X\}$ and $A_{\alpha+} = \{x \mid m_A(x) > \alpha,\ x \in X\}$ for $\alpha \in [0, 1]$. The level-value set of A is defined by $L_A = \{\alpha \mid m_A(x) = \alpha$ for some $x \in X\}$, i.e., the range of membership function m_A. For any crisp (not fuzzy) subset B of X, we use αB to denote the fuzzy set having membership function $m_B = \alpha \chi_B$ for any $\alpha \in [0, 1]$.

Decomposition Theorem I.

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha = \bigcup_{\alpha \in (0,1]} \alpha A_\alpha.$$

Decomposition Theorem II.

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_{\alpha+} = \bigcup_{\alpha \in (0,1]} \alpha A_{\alpha+} = \bigcup_{\alpha \in (0,1)} \alpha A_{\alpha+}.$$

Decomposition Theorem III.

$$A = \bigcup_{\alpha \in L_A} \alpha A_\alpha.$$

Regarding fuzzy numbers as special fuzzy subsets of $\mathbb{R}$, these decomposition theorems are also available for fuzzy numbers. Unfortunately, none of them can be used to define a total ordering on $\mathcal{N}_F$ since they identify a fuzzy number by uncountably many real-valued parameters generally and, therefore, the lexicography cannot be used any more. Thus, establishing a new decomposition theorem, which identifies any fuzzy number by only countably many real-valued parameters, for fuzzy numbers is essential.

Theorem 3 *(Decomposition Theorem IV). Let A be a fuzzy set with membership function m_A and S be a given upper dense sequence in* $(0,1]$. *Then*

$$A=\bigcup_{\alpha\in S}\alpha A_\alpha.$$

Proof. On one hand, since $S\subseteq[0,1]$, we have $\bigcup_{\alpha\in S}\alpha A_\alpha\subseteq\bigcup_{\alpha\in[0,1]}\alpha A_\alpha=A$. On the other hand, we need to show that

$$m_A(x)\leqslant\sup_{\alpha\in S}\alpha\chi_{A_\alpha}(x)$$

for every $x\in X$. In fact, for each given $x\in X$, by Decomposition Theorem II,

$$m_A(x)=\sup_{\alpha\in(0,1)}\alpha\chi_{A_{\alpha+}}(x)=\sup_{\alpha\in(0,m_A(x))}\alpha\chi_{A_{\alpha+}}(x).$$

For each $\alpha\in(0,m_A(x))$, since S is also lower dense in $(0,1]$, we may find a real number $\beta_\alpha\in(\alpha,m_A(x))\cap S$ so that

$$\alpha\chi_{A_{\alpha+}}(x)<\beta_\alpha\chi_{A_{\alpha+}}(x)=\beta_\alpha=\beta_\alpha\chi_{A_{\beta_\alpha}}(x)\leqslant\sup_{\beta\in S}\beta\chi_{A_\beta}(x).$$

Thus, taking the supremum with respect to $\alpha\in(0,m_A(x))$, we obtain

$$m_A(x)=\sup_{\alpha\in(0,m_A(x))}\alpha\chi_{A_{\alpha+}}(x)\leqslant\sup_{\beta\in S}\beta\chi_{A_\beta}(x)=\sup_{\alpha\in S}\alpha\chi_{A_\alpha}(x).$$

The proof is now complete. □

6. Defining total orderings on the set of all fuzzy numbers

Decomposition Theorem IV established in Section 5 identifies any fuzzy number by using only countably many real-valued parameters. It provides us with a powerful tool for defining total orderings on the set of all fuzzy numbers, $\mathcal{N}_F$, by using the extended lexicography.

Given an upper dense sequence $S=\{\alpha_i\mid i=1,2,\ldots\}$ in $(0,1]$, for any given fuzzy number $\tilde{e}\in\mathcal{N}_F$, from Section 2 we know that the α-cut of $\tilde{e}$ at each α_i, $i=1,2,\ldots$, is a closed interval. Denote this interval by $[a_i,b_i]$, and let $c_{2i-1}=a_i+b_i$ (the twice of the middle of the interval) and $c_{2i}=b_i-a_i$ (the length of the interval), $i=1,2,\ldots$. By Decomposition Theorem IV, these countably many parameters $\{c_j\mid j=1,2,\ldots\}$ identify the fuzzy number. Using these parameters, we define a relation on $\mathcal{N}_F$ as follows.

Definition 3. Let $\tilde{e}$ and $\tilde{f}$ be two fuzzy numbers. For given upper dense sequence $S=\{\alpha_i\mid i=1,2,\ldots\}$ in $(0,1]$, we use $c_j(e)$ and $c_j(f)$ to denote above-mentioned c's for $\tilde{e}$ and $\tilde{f}$ respectively. We say that $\tilde{e}=\tilde{f}$ if and only if their α-cuts at α_i are equal to each other, that is, $e_{\alpha_i}=f_{\alpha_i}$, for all $i=1,2,\ldots$; we say that $\tilde{e}\prec\tilde{f}$ if and only if $\tilde{e}=\tilde{f}$ is not true and there exists a positive integer j such that $c_j(e)<c_j(f)$ and $c_i(e)=c_i(f)$ for all positive integers $i<j$; we say that $\tilde{e}\preccurlyeq\tilde{f}$ if and only if $\tilde{e}\prec\tilde{f}$ or $\tilde{e}=\tilde{f}$.

Intuitively, integer j in Definition 3 is the smallest positive integer such that $c_j(e)\neq c_j(f)$ and the ordering relation of these two fuzzy numbers is determined by only the inequality between $c_j(e)$ and $c_j(f)$. To be convenient in the proof of the next theorem, we may use $j=\infty$ to denote the case that there is no positive integer j such that $c_j(e)\neq c_j(f)$, that is, the case $\tilde{e}=\tilde{f}$. We also define $i<\infty$ for any positive integer i and $\min(\infty,\infty)=\infty$.

Theorem 4. *Relation $\preccurlyeq$ is a total ordering on $\mathscr{N}_F$.*

Proof. We prove this conclusion by 4 steps as follows.

(1) Showing the reflexivity of relation $\preccurlyeq$ is trivial.

(2) To show the antisymmetry of $\preccurlyeq$, let $\tilde{e}$, and $\tilde{f}$ be two fuzzy numbers satisfying $\tilde{e} \preccurlyeq \tilde{f}$ and $\tilde{f} \preccurlyeq \tilde{e}$. Assuming that $\tilde{e} \neq \tilde{f}$, we have both $\tilde{e} \prec \tilde{f}$ and $\tilde{f} \prec \tilde{e}$. From the former, we can find j_1, such that $c_{j_1}(e) < c_{j_1}(f)$ and $c_j(e) = c_j(f)$ for all positive integers $j < j_1$; from the latter, we can find j_2, such that $c_{j_2}(f) < c_{j_2}(e)$ and $c_j(f) = c_j(e)$ for all positive integers $j < j_2$. Then, j_1 and j_2 must be the same, denoted as j_0. But holding both $c_{j_0}(e) < c_{j_0}(f)$ and $c_{j_0}(f) < c_{j_0}(e)$ is impossible and, therefore, the above assumption is wrong, i.e., we must have $\tilde{e} = \tilde{f}$.

(3) To show that $\preccurlyeq$ is transitive, let $\tilde{e}$, $\tilde{f}$, and $\tilde{g}$ be three fuzzy numbers satisfying $\tilde{e} \preccurlyeq \tilde{f}$ and $\tilde{f} \preccurlyeq \tilde{g}$. From $\tilde{e} \preccurlyeq \tilde{f}$, we can find j_1, which may be ∞, such that $c_{j_1}(e) < c_{j_1}(f)$ and $c_j(e) = c_j(f)$ for all positive integers $j < j_1$; from $\tilde{f} \preccurlyeq \tilde{g}$, we can find j_2, which may be ∞, such that $c_{j_2}(f) < c_{j_2}(g)$ and $c_j(f) = c_j(g)$ for all positive integers $j < j_2$. Taking $j_0 = \min(j_1, j_2)$, we have $c_{j_0}(e) < c_{j_0}(g)$ and $c_j(e) = c_j(g)$ for all positive integers $j < j_0$, that is, $\tilde{e} \preccurlyeq \tilde{g}$. This means that relation $\preccurlyeq$ is transitive.

Up to here, we have shown that $\preccurlyeq$ is a partial ordering on $\mathscr{N}_F$. Furthermore, we need to show that any two fuzzy numbers are comparable according to relation $\preccurlyeq$.

(4) For any two fuzzy number $\tilde{e}$ and $\tilde{f}$, they are either $\tilde{e} = \tilde{f}$, or this equality is not true, i.e., $\tilde{e} \neq \tilde{f}$. In the latter case, there are some integers j such that $c_j(e) \neq c_j(f)$. Let $J = \{j \mid c_j(e) \neq c_j(f)\}$. Then J is lower bounded and, therefore, according to the Well-Ordering Property [18], J has a unique smallest element, denoted by j_0. Thus, we have $c_j(e) = c_j(f)$ for all positive integers $j < j_0$, and either $c_{j_0}(e) < c_{j_0}(f)$ or $c_{j_0}(e) > c_{j_0}(f)$, that is, either $\tilde{e} \prec \tilde{f}$ or $\tilde{f} \prec \tilde{e}$ in this case. So, for these two fuzzy numbers, either $\tilde{e} \preccurlyeq \tilde{f}$ or $\tilde{f} \preccurlyeq \tilde{e}$. This means that partial ordering $\preccurlyeq$ is a total ordering on $\mathscr{N}_F$.

The proof is now complete. □

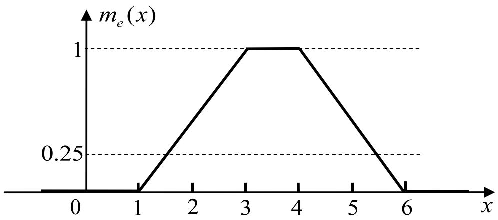
Similar to the case of total orderings on the real line $(-\infty, \infty)$ and the total orderings on sets consisting of special types of fuzzy numbers shown in Section 3, infinitely many different total orderings on $\mathscr{N}_F$ can be defined. Even using a given upper dense sequence in $(0, 1]$, there are still infinitely many different ways to determine a total ordering on $\mathscr{N}_F$. A notable fact is that each of them is consistent with the natural ordering on the set of all real numbers. This can be regarded as a fundamental requirement for any practice ordering method on the set of all fuzzy numbers. Nevertheless, it is intuitive and convenient to adopt sequence $S_b = \{d_{bi} \mid i = 1, 2, \ldots\}$ shown in Example 1 and use the way presented in Definition 3 as the default. This is because the core, at which the membership degree is 1 everywhere, of fuzzy numbers is used in the first step of the comparison, and is the most important factor to identifying and ranking fuzzy numbers.

We can also see that, in Example 2, though sequences S_r and S'_r are different, by Definition 3, the defined two total orderings on $\mathscr{N}_F$ are the same.

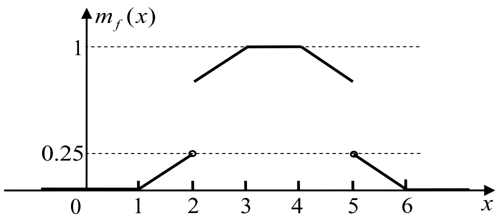
In comparison with the Hurwicz criterion in economic decision [5,11,12], each term $c_{2i-1} = a_i + b_i$ $(i = 1, 2, \ldots)$ shown above is just the twice of weighted average with weight $w = 0.5$ in his model. It is the mutualism (neither pessimism nor optimism) of consideration. In Hurwicz's economic decision model, since only a ranking on the set of all closed intervals is required, one index is sufficient for this purpose. However, we now want to define a total ordering on the set of all fuzzy numbers, only taking one index to order intervals is not sufficient. As one of the reference systems, we should take another index $c_{2i} = b_i - a_i$ after the first criterion as a supplementary one for each $i = 1, 2, \ldots$. Of course, selecting any two weighted averages with different weights from Hurwicz's model as indexes and assigning their order to form a reference system also work in the above-mentioned approach for defining a total ordering on $\mathscr{N}_F$.

7. Ranking fuzzy numbers by total orderings

The following two examples show how the total orderings work for ranking fuzzy numbers. In fact, any total ordering is a special ranking. Unlike the previously existing ranking methods, by which people can always cite some different fuzzy numbers with the same rank and, therefore, they cannot be ordered, the proposed total ordering can order any given fuzzy numbers.



(a) The membership function of fuzzy number $\tilde{e}$



(b) The membership function of fuzzy number $\tilde{f}$

Fig. 1. The membership function of fuzzy numbers in Example 3.

Example 3. Let $\tilde{e}$ and $\tilde{f}$ be fuzzy numbers with membership functions

$$m_e(x) = \begin{cases} 0.5x - 0.5 & \text{if } x \in [1, 3) \\ 1 & \text{if } x \in [3, 4] \\ 3 - 0.5x & \text{if } x \in (4, 6] \\ 0 & \text{otherwise} \end{cases}$$

and

$$m_f(x) = \begin{cases} 0.25x - 0.25 & \text{if } x \in [1, 2) \\ 0.25x + 0.25 & \text{if } x \in [2, 3) \\ 1 & \text{if } x \in [3, 4] \\ 2 - 0.25x & \text{if } x \in (4, 5] \\ 1.5 - 0.25x & \text{if } x \in (5, 6] \\ 0 & \text{otherwise} \end{cases}$$

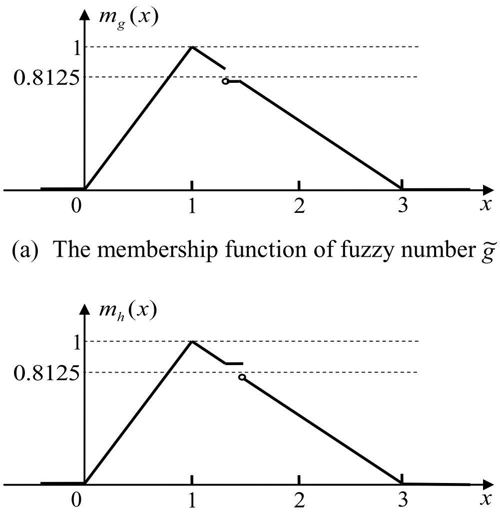
shown in Fig. 1 respectively. The total ordering $\preccurlyeq$ defined by using upper dense sequence S_b given in Example 1 and the way shown in Definition 3 are now adopted. We have $c_1(e) = c_1(f) = 7$, $c_2(e) = c_2(f) = 1$, $c_3(e) = c_3(f) = 7$, $c_4(e) = c_4(f) = 3$, $c_5(e) = c_5(f) = 7$, but $c_6(e) = 4$ while $c_6(f) = 3$, which correspond to the length of the intervals as the respect α-cut of $\tilde{e}$ and $\tilde{f}$ at $\alpha = 1/4$. So, $\tilde{f} \prec \tilde{e}$. It should be noted that the defined total ordering depends on the choice of the upper dense sequence. For instance, if we choose $(1, 1/2, 3/4, 1/4, 7/8, 5/8, \ldots)$ as the upper dense sequence, then a different conclusion $\tilde{e} \prec \tilde{f}$ will be obtained. This is similar to the fact that choosing different ranking indices people may obtain different conclusion on the rank of some fuzzy numbers.

Example 4. Let $\tilde{g}$ and $\tilde{h}$ be fuzzy numbers with membership functions

$$m_g(x) = \begin{cases} x & \text{if } x \in [0, 1] \\ 0.8 & \text{if } x \in (1.3, 1.4] \\ 1.5 - 0.5x & \text{if } x \in (1, 1.3] \cup (1.4, 3] \\ 0 & \text{otherwise} \end{cases}$$

and

$$m_h(x) = \begin{cases} x & \text{if } x \in [0, 1] \\ 0.85 & \text{if } x \in (1.3, 1.4] \\ 1.5 - 0.5x & \text{if } x \in (1, 1.3] \cup (1.4, 3] \\ 0 & \text{otherwise} \end{cases}$$



(a) The membership function of fuzzy number $\tilde{g}$

(b) The membership function of fuzzy number $\tilde{h}$

Fig. 2. The membership function of fuzzy numbers in Example 4.

shown in Fig. 2 respectively. We use the same total ordering $\preccurlyeq$ adopted at the beginning of Example 3, and obtain that $c_j(g) = c_j(h)$ for $j = 1, 2, \ldots, 14$, but $c_{15}(g) = 1.925$ while $c_{15}(h) = 2.025$. So, $\tilde{g} \prec \tilde{h}$.

When the above default total ordering is adopted for ranking fuzzy numbers, restricted on the set of all rectangular fuzzy numbers, only parameters c_1 and c_2 are concerned; restricted on the set of all triangular fuzzy numbers, only parameters c_1, c_3, and c_4 are concerned; while restricted on the set of all trapezoidal fuzzy numbers, only parameters c_1, c_2, c_3, and c_4 are concerned. From here, we can also see the rationale of favoring an upper dense sequence over a lower dense sequence, for defining a total ordering on $\mathscr{N}_F$.

The total ordering discussed above can also be used as a supplementary means for ordering fuzzy numbers when other intuitive ranking methods fail. The following example shows how the above total ordering works when the ranking method based on the centroid [20] fails.

Example 5. Let $\tilde{s}$ and $\tilde{t}$ be triangular fuzzy numbers with membership functions

$$m_s(x) = \begin{cases} \frac{1}{2}(x-1) & \text{if } x \in [1,3] \\ \frac{1}{2}(5-x) & \text{if } x \in (3,5] \\ 0 & \text{otherwise} \end{cases}$$

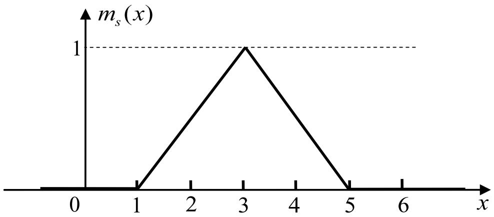
and

$$m_t(x) = \begin{cases} \frac{1}{3}(x-1) & \text{if } x \in [1,4] \\ 0 & \text{otherwise} \end{cases}$$

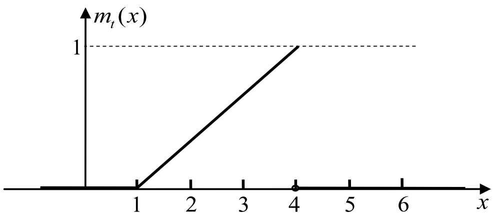
shown in Fig. 3 respectively. Fuzzy numbers $\tilde{s}$ and $\tilde{t}$ have the same centroid $(3, \frac{1}{3})$ and, therefore, they have the same rank, i.e., they cannot be ordered by using the method of centroid shown in [4,20]. However, using the total ordering introduced in Section 6 of this work, we have $\tilde{s} \prec \tilde{t}$ since $c_1(s) = 3 < 4 = c_1(t)$.

8. Conclusions

On the contrary to previous studies on ranking fuzzy numbers in literature, which define total orderings either on a quotient space according to some equivalence relation for fuzzy numbers or on a set consisting of only some special types of fuzzy numbers, based on a given upper dense sequence in (0, 1] and a new decomposition theorem for fuzzy sets, we present a method for defining total orderings on the set of all fuzzy numbers directly. Some existing total orderings on special types of fuzzy numbers can be regarded as restrictions of our total orderings.



(a) The membership function of fuzzy number $\tilde{s}$



(b) The membership function of fuzzy number $\tilde{t}$

Fig. 3. The membership function of fuzzy numbers in Example 5.

Similar to the situation of other total ordered infinite sets, even if the upper dense sequence is fixed, we still have infinitely many different ways to define a total ordering on the set of all fuzzy numbers. In this work, we suggest a default of choosing an upper dense sequence and a common way used to order closed intervals for defining a total ordering on the set of all fuzzy numbers with geometric intuition according to the location of their membership curves.

The total orderings introduced and discussed in this work are consistent with the natural ordering of real numbers and, therefore, is a real generalization of the total ordering on the set of all real numbers to the set of all fuzzy numbers. This method can order fuzzy numbers, either alone or as a supplementary means with other ranking methods, and may be adopted in decision making with fuzzy information.

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