Arithmetic: Past Revisited

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Ambitious beginnings

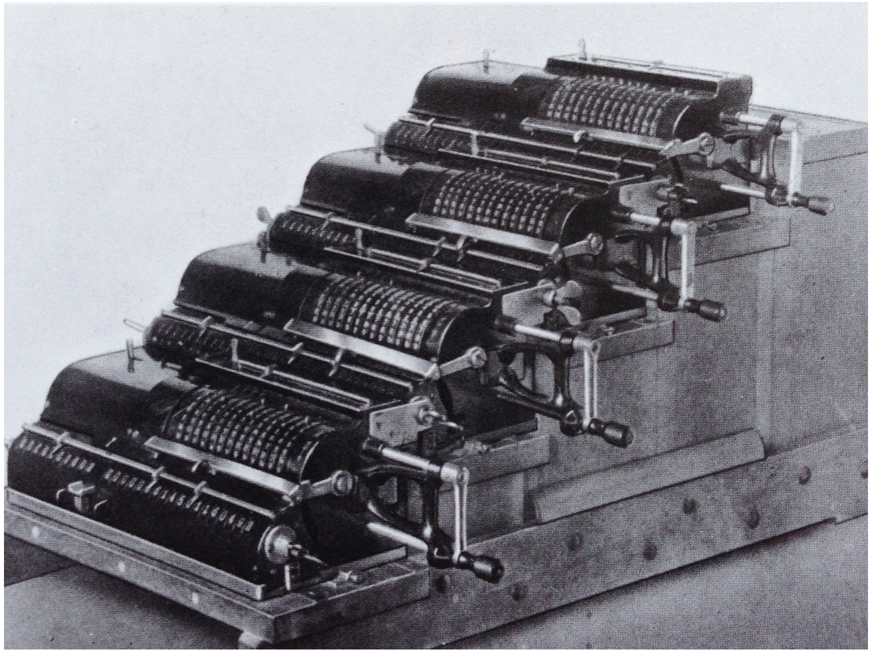
Two pillars of digital arithmetic: signed-digit and carry-save

DCL and Illiacs at the University of Illinois Urbana-Champaign

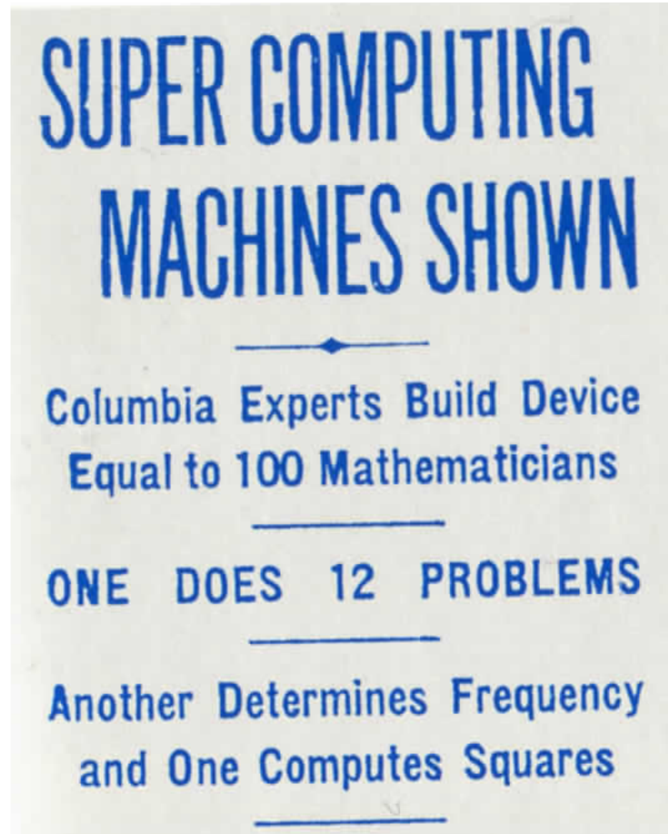
Antonin Svoboda and ARITH-4, Santa Monica, 1978.

Two specimens from the past

Early attempt at arithmetic chaining - A. J. Thompson integrating and differencing machine made of four connected calculators. Unfortunately (for Mr. Thompson) Moore's Law did not apply here.



Enthusiasm for fast arithmetic rising - New York World, March 1, 1920.



SUPER COMPUTING MACHINES SHOWN

Columbia Experts Build Device Equal to 100 Mathematicians

ONE DOES 12 PROBLEMS

Another Determines Frequency and One Computes Squares

Amusing and, perhaps, silly today?

That's the fate of most ideas and their implementations after years pass by.

But there are some ”old” ideas in arithmetic that don't share such a fate ....

Amazingly modern work on signed-digit arithmetic: Early 18th century

John Colson, Vicar then Lucasian Professor of Mathematics at Cambridge (1739-1760), translated Newton's works, Fellow of Royal Society.

I. *A ſhort Account of Negativo-affirmative Arithmetick, by Mr.* John Colſon, F. R. S.

Philosophical Transactions of the Royal Society, Vol.34 (1726-27)

[In May 1726, the London newspapers announced the arrival of Francois Marie Arouet, the renowned French dramatist and poet, known as Voltaire. Surely not to practice signed-digit arithmetic - after a brief free stay in Bastille, he was exiled for behavior that irritated nobility.]

THE Uſefulneſs of this Arithmetick conſiſts in this, that it performs all the Operations with more Eaſe and Expedition than the common Affirmative Arithmetick, eſpecially in large Numbers: And it differs from the common Arithmetick chiefly in this, that it admits of Negative Figures promiſcuouſly with the affirmative. Theſe negative Figures are diſtinguiſh'd from the Affirmative, by the Sign - placed over them.

Thus $3\bar{7}09\bar{2}\bar{8}65\bar{7}3\bar{9}\bar{6}\bar{1}47\bar{2}$ is one of theſe Numbers, which may be converted into its Equivalent common Number 2308726432039468, in this manner:

$$
\begin{array}{r}
3009006503000470 \\
\underline{0700280070961002} \\
2308726432039468
\end{array}
$$

Colson proposes a radix-10 left-to-right conversion algorithm of SD number into conventional form.

Let p_i and n_i denote positive and negative digits (”Figures”). The conversion rules are based on pairs of digits:

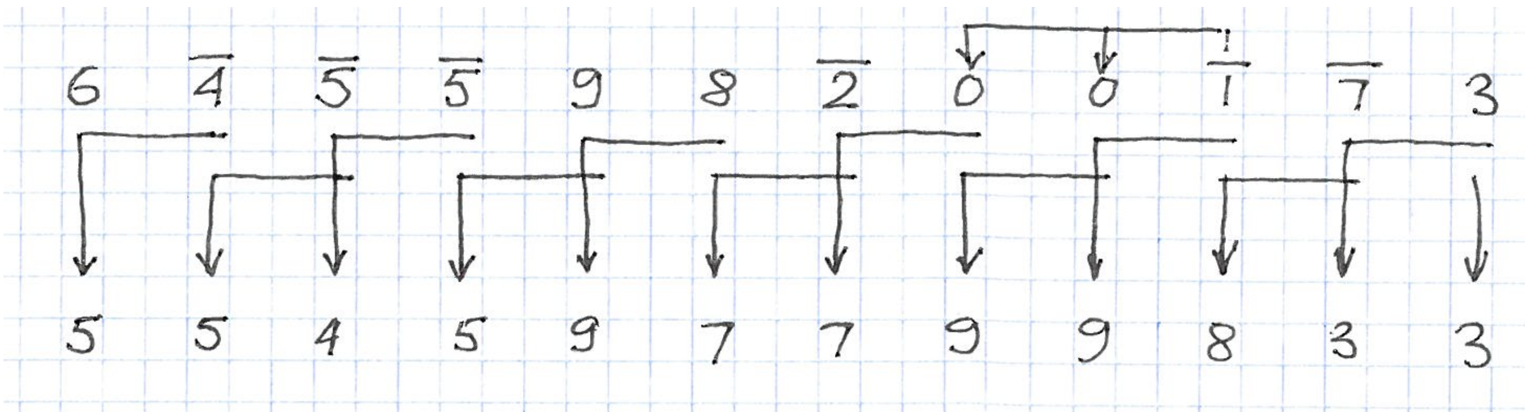
1. $p_i n_{i-1} \Rightarrow (p_i - 1)$
2. $n_i p_{i-1} \Rightarrow (10 - |n_i|)$
3. $n_i n_{i-1} \Rightarrow (9 - |n_i|)$
4. All other digits must remain unchanged.

Sign of zero (”Cypher”) is the sign of the next non-zero digit - requires propagation.

Pairs of non-zero digits are converted in parallel.

Example:

Signed-digit input



Conventional output

Rules:

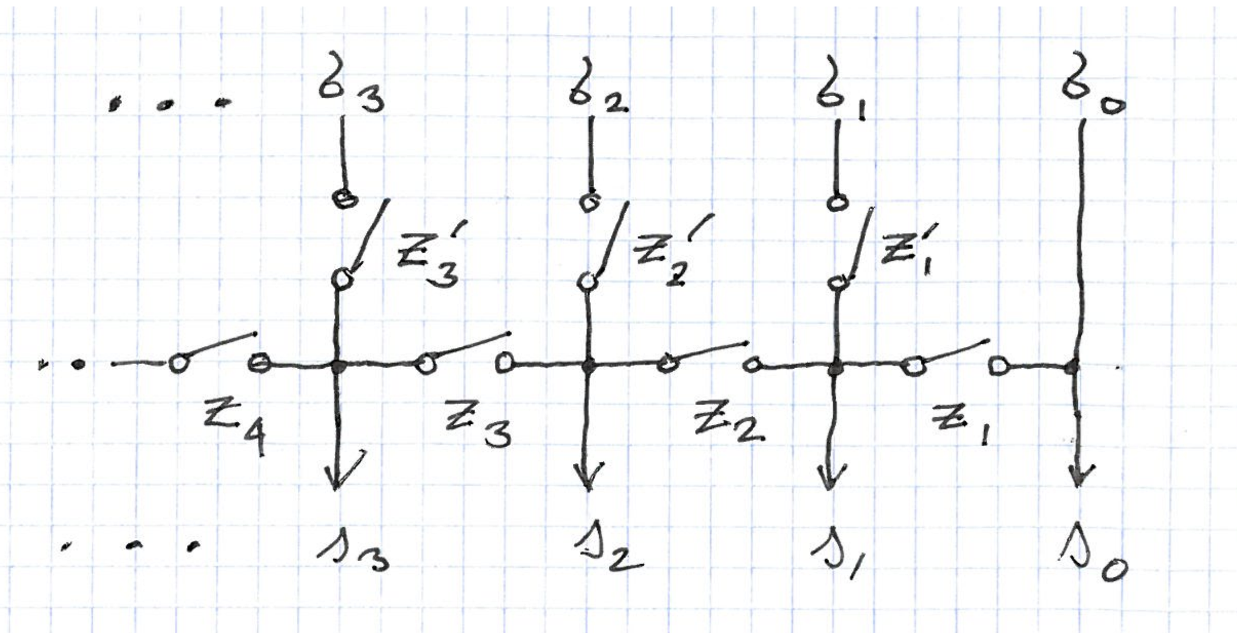
$$p_i n_{i-1} \Rightarrow (p_i - 1); \, n_i p_{i-1} \Rightarrow (10 - |n_i|); \, n_i n_{i-1} \Rightarrow (9 - |n_i|)$$

Sign Propagation

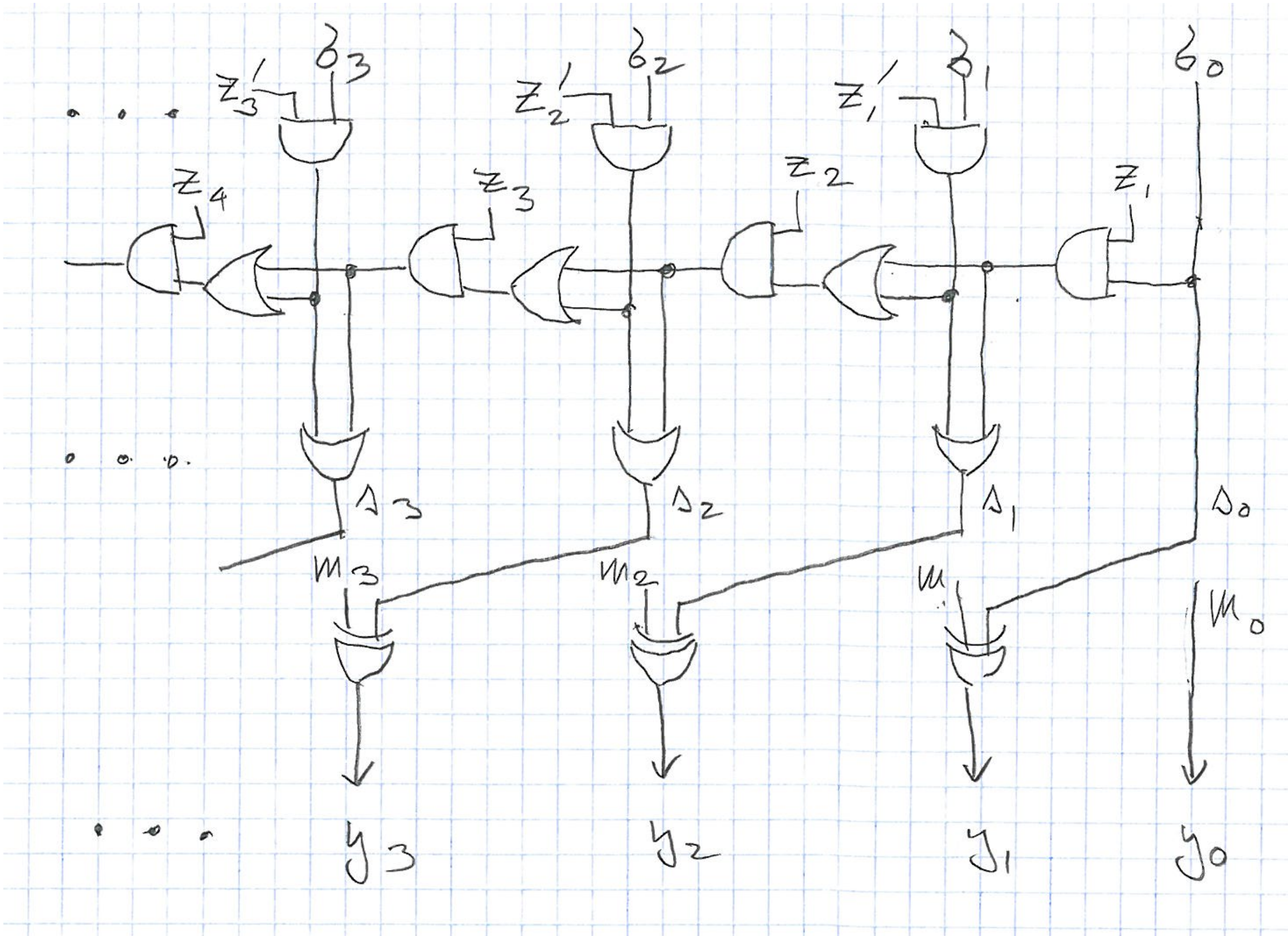
Input digit: $x_i = (\sigma_i, m_i)$, sign $\sigma_i =$(0 if positive, 1 if negative),

Magnitude $m_i \in \{0, 1, \ldots, r-1\}$, zero signal $z_i = (m_i = 0)$

Switch network for sign propagation



SD–to–Conventional Radix-2 Converter Circuit



Throwing out all the large digits

$$\{9,\ 8,\ 7,\ 6\} \to \{1\bar{1},\ 1\bar{2},\ 1\bar{3},\ 1\bar{4}\}$$

5 "esteemed" to be large or small depending on context

Rules: 1. $SL \Rightarrow (S+1)$; 2. $LL \Rightarrow -(9-L)$; 3. $LS \Rightarrow -(10-L)$

$$(m)\,387916407953,\&c. = (m)\ 4\bar{1}\bar{2}\bar{1}2\bar{4}41\bar{2}\bar{1}53,\&c.$$

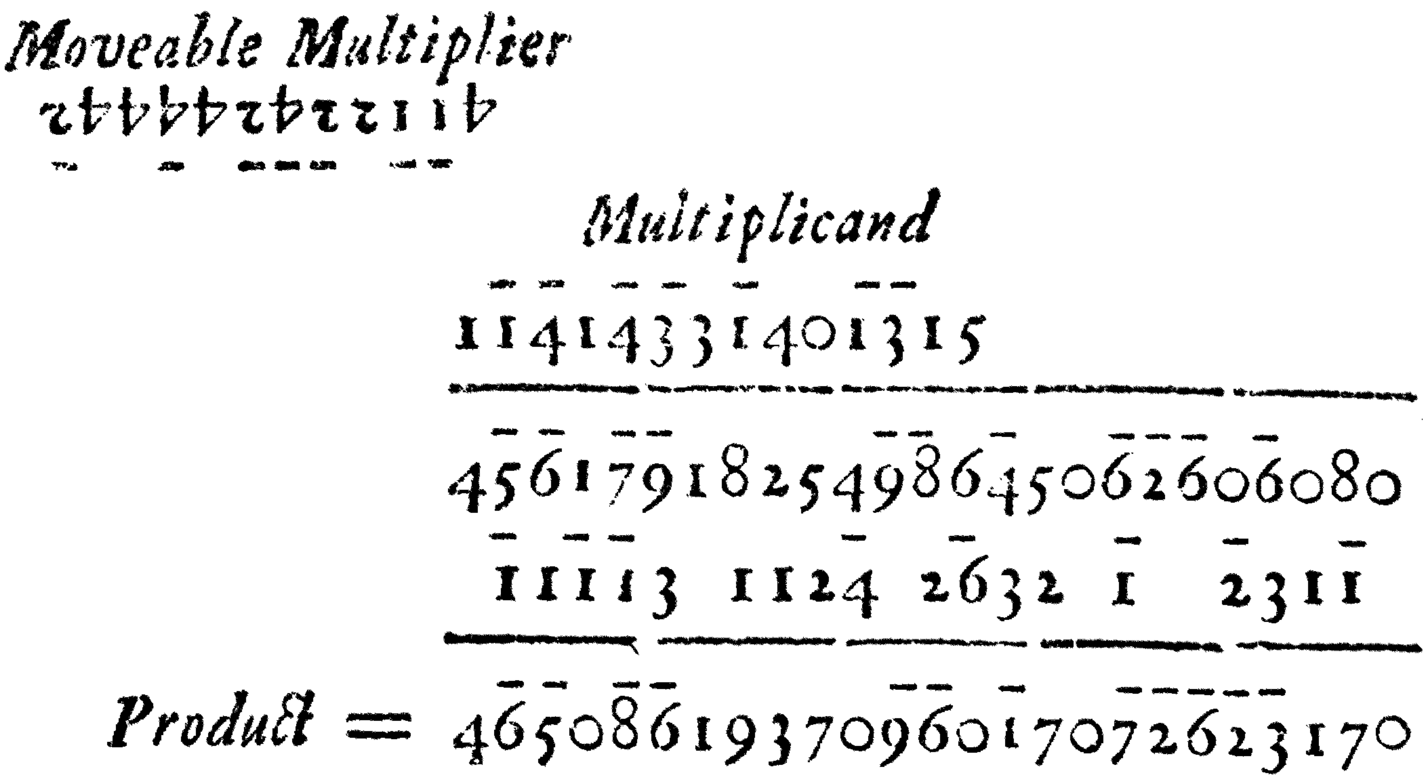
- Index m *"stands for some Integer, expressing the Distance of the first Figure from the Place of Units; which Integer is either affirmative or negative;"*

$\&c.$ denotes "interminate" (approximate) number, *"... and all the rest (whether finite or infinite in Number, whether known or unknown)"*

Multioperand SD Addition

(m)	$2\bar{1}5\,\bar{3}\bar{1}\bar{4}0\bar{4}3\;\bar{1}\bar{2}\bar{1}3$, &c.
(m)	$50\bar{4}\bar{2}0\,3\bar{1}\bar{4}255\bar{1}\bar{2}$, &c.
$(m-1)$	$4\bar{3}10\bar{2}\bar{3}10\bar{2}\bar{4}\bar{1}\bar{3}$, &c.
$(m-2)$	$5\bar{1}3\bar{4}\bar{2}1\bar{1}03\bar{2}1$, &c.
$(m-3)$	$2\bar{1}30\bar{4}2\bar{1}03\bar{2}$, &c.
$(m-4)$	$1\bar{3}\bar{2}021\bar{2}2\bar{4}$, &c.
$(m-5)$	$13\bar{2}\bar{2}43\bar{1}5$, &c.
$(m+1)$	$1\bar{3}33\bar{3}2141\bar{1}\bar{3}\bar{4}3\bar{1}\bar{2}$, &c.

Multiplication - Left to Right



- Convert operands to ”small” signed-digit form

- Perform pairwise multiplication of overlapped digits and add:

$4 \times 1 = 4$

$4 \times \bar{1} + \bar{1} \times \bar{1} = \bar{5}$

$4 \times \bar{4} + \bar{1} \times \bar{1} + \bar{1} \times 1 = \bar{1}\bar{6}$

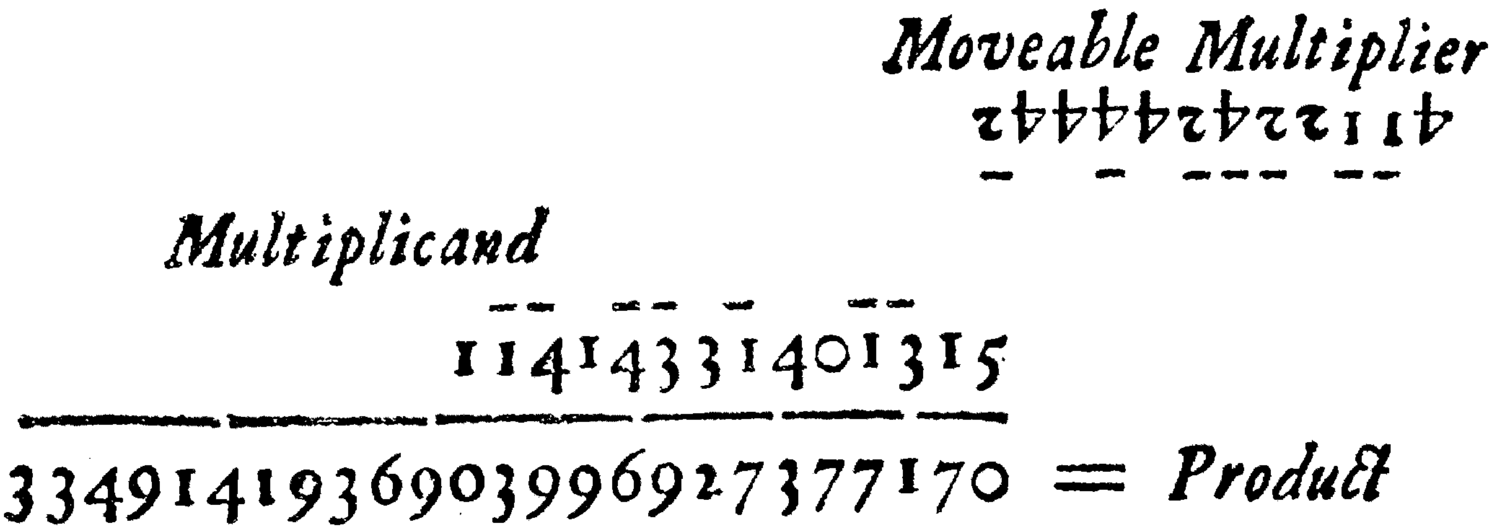
etc.

- Shift the multiplier one position right after obtaining sum of digit products

- Repeat for all digit positions

- To obtain an approximate product, compute one more place

Multiplication - Right to Left



- Shift the multiplier one position left after obtaining a product digit

- Perform pairwise multiplication of overlapped digits, form their sum and add carry immediately from the position to the right

Signed-Digit Division

- Operands first reduced to small digit form; form divisor multiples

- Quotient digit selection by comparing residual with divisor multiples $(1 \times d, \ldots, 5 \times d)$:

TABLE of Multiples.

	(n) $2\bar{2}4\bar{4}10\bar{3}13\bar{4}5\bar{1}$, &c.
1	(n) $2\bar{2}4\bar{4}10\bar{3}13\bar{4}5\bar{1}$, &c.
2	(n) $4\bar{3}\bar{3}22\bar{1}42530\bar{2}$, &c.
3	(n) $551\bar{2}3\bar{1}14\bar{2}\bar{1}5\bar{3}$, &c.
4	(n+1) $1\bar{3}3444\bar{1}\bar{2}51\bar{4}0\bar{4}$, &c.
5	(n+1) $1\bar{1}2\bar{2}05\bar{1}\bar{4}\bar{4}3245$, &c.

$$\textit{Quotient} = (m)\ 31\bar{3}150\bar{3}42\bar{3}5\bar{4},\ \&c.$$

$$\textit{Dividend} = (m + \mathrm{n} + 1)\ \bar{1}\bar{4}440\bar{4}\bar{3}\bar{2}\bar{4}\bar{2}222,\ \&c.$$

$$\bar{5}\bar{5}\bar{1}2\bar{3}1\bar{1}\bar{4}21\bar{5}3,\ \&c.$$

$$1313\bar{2}\bar{4}203\bar{3}5,\ \&c.$$

$$\bar{2}2\bar{4}4\bar{1}03\bar{1}\bar{3}4\bar{5},\ \&c.$$

$$\bar{1}5\bar{2}\bar{3}\bar{3}\bar{4}5\bar{1}010,\ \&c.$$

$$551\bar{2}3\bar{1}14\bar{2}\bar{1},\ \&c.$$

$$3\bar{3}5\bar{1}404\bar{1}\bar{1},\ \&c.$$

$$\bar{2}2\bar{4}4\bar{1}03\bar{1}\bar{3},\ \&c.$$

$$1\bar{1}1331\bar{3}\bar{2}\bar{4},\ \&c.$$

$$\bar{1}1\bar{2}20\bar{5}144,\ \&c.$$

$$\bar{1}53\bar{4}\bar{2}20,\ \&c.$$

$$551\bar{2}3\bar{1},\ \&c.$$

Colson concludes:

And this may ſuffice for a ſhort Summary of Negativo-affirmative Arithmetick, as to the ordinary Operations of Addition, Subtraction, Multiplication, and Diviſion. What improvements may be had from hence in the Extraction of Roots, whether of pure or affected Equations, I ſhall leave to future inquiry.

"I have contrived an Instrument, call'd Abacus, or the Counting Table, which I hope shortly to communicate to the inquisitive in these matters and by which all long Calculations may be very much facilitated."

1840 - Signed-Digit French Way: Cauchy

In Extrait No. 105 *Calculus Numérique. - Sur le moyens d'eviter les erreurs dans les calculs numériques*, Comptes Rendus de L'Academie, 16 Nov. 1840:

Cauchy describes signed-digit representation and recoding to limit the decimal digit set to $\{-4, \ldots, 5\}$

$1\bar{1} = 9 \quad 1\bar{2}1 = 81$

$10\bar{2}\bar{4}5\bar{3}12\bar{4}\bar{2} = 976471158$

and comments that such a signed-digit set simplifies all operations.

In particular, multiples of $2,\ 3,\ 4 = 2 \times 2$ and $5 = 10/2$ suffice in multiplication: that is, doubling, tripling and taking half.

Example of multiplication:

$8\,2\,5\,6 = 1\,\bar{2}\,3\,\bar{4}\,\bar{4} \qquad 9\,9\,7\,8 = 1\,0\,0\,\bar{2}\,\bar{2}$

Partial products with simple multiples

$$
\begin{array}{ccccccccc}
 & & & & \bar{2} & 4 & \bar{5} & \bar{1} & \bar{2} \\
 & & & \bar{2} & 4 & \bar{5} & \bar{1} & \bar{2} & \\
1 & \bar{2} & 3 & \bar{4} & \bar{4} & & & & \\
\hline
1 & \bar{2} & 2 & 4 & \bar{2} & \bar{2} & 4 & \bar{3} & \bar{2}
\end{array}
$$

Simple fractions to decimal fractions

$$
\begin{aligned}
\frac{1}{7} &= 0.\,1\,4\,2\,8\,5\,7\,1\,4\,2\,8\,5\,7\ldots \\
&= 0.\,1\,4\,3\,\bar{1}\,\bar{4}\,\bar{3}\,1\,4\,3\,\bar{1}\,\bar{4}\,\bar{3}\ldots
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{11} &= 0.\,0\,9\,0\,9\,0\,9\,0\,9\,0\,9\,0\,9\ldots \\
&= 0.\,1\,\bar{1}\,1\,\bar{1}\,1\,\bar{1}\,1\,\bar{1}\,1\,\bar{1}\,1\,\bar{1}\ldots
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{13} &= 0.\,0\,7\,6\,9\,2\,3\,0\,7\,6\,9\,2\,3\ldots \\
&= 0.\,1\,\bar{2}\,\bar{3}\,\bar{1}\,2\,3\,1\,\bar{2}\,\bar{3}\,\bar{1}\,2\,3\ldots
\end{aligned}
$$

Applies binomial formula for squares and cubes:

$$((1\ 0)\ (1\ 3))^2 \ = \ (1\ 0)^2 * 10^4 + 2 * (1\ 0) * (1\ 3) + (1\ 3)^2 \ = \ (1\ 0\ 2\ 6\ 1\ 6\ 9)$$

$$((1\ 0)\ (\bar{1}\ \bar{3}))^2 \ = \ (1\ 0\ \bar{2}\ \bar{6}\ 1\ 6\ 9)$$

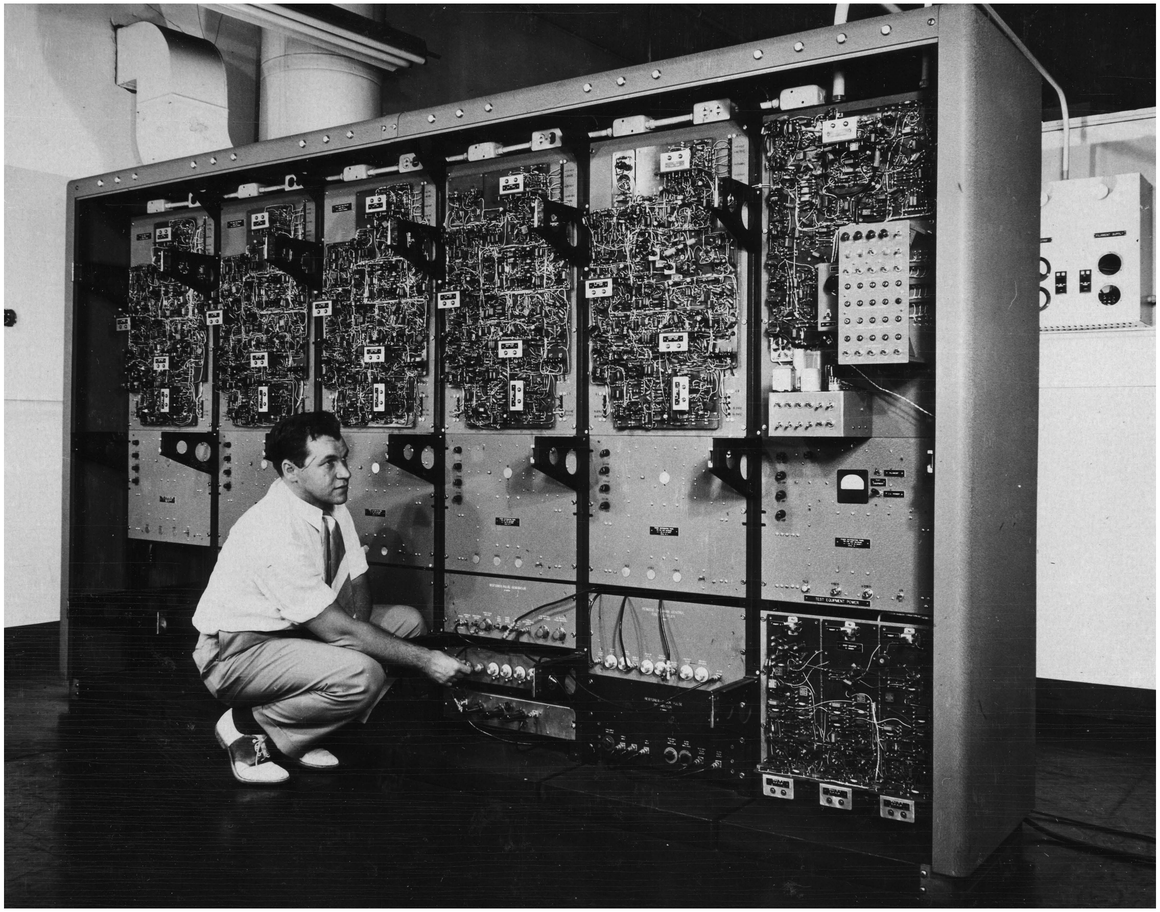
$$(987)^2 \ = \ 9\ 7\ 4\ 1\ 6\ 9$$

$$((1\ 0)\ (0\ 6))^3 \ = \ 1\ 0\ 1\ 8\ 1\ 0\ 8\ 2\ 1\ 6$$

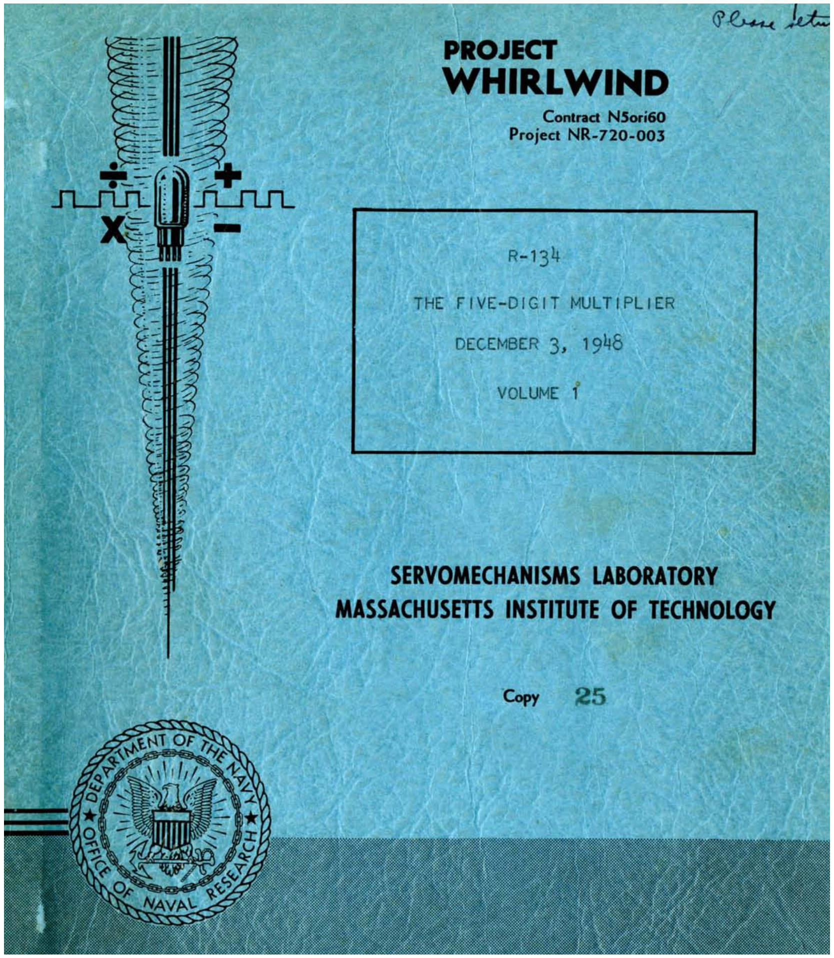
$$((1\ 0)\ (0\ \bar{6}))^3 \ = \ 1\ 0\ \bar{1}\ \bar{8}\ 1\ 0\ 8\ \bar{2}\ \bar{1}\ \bar{6}$$

$$(994)^3 \ = \ 9\ 8\ 2\ 1\ 0\ 7\ 7\ 8\ 4$$

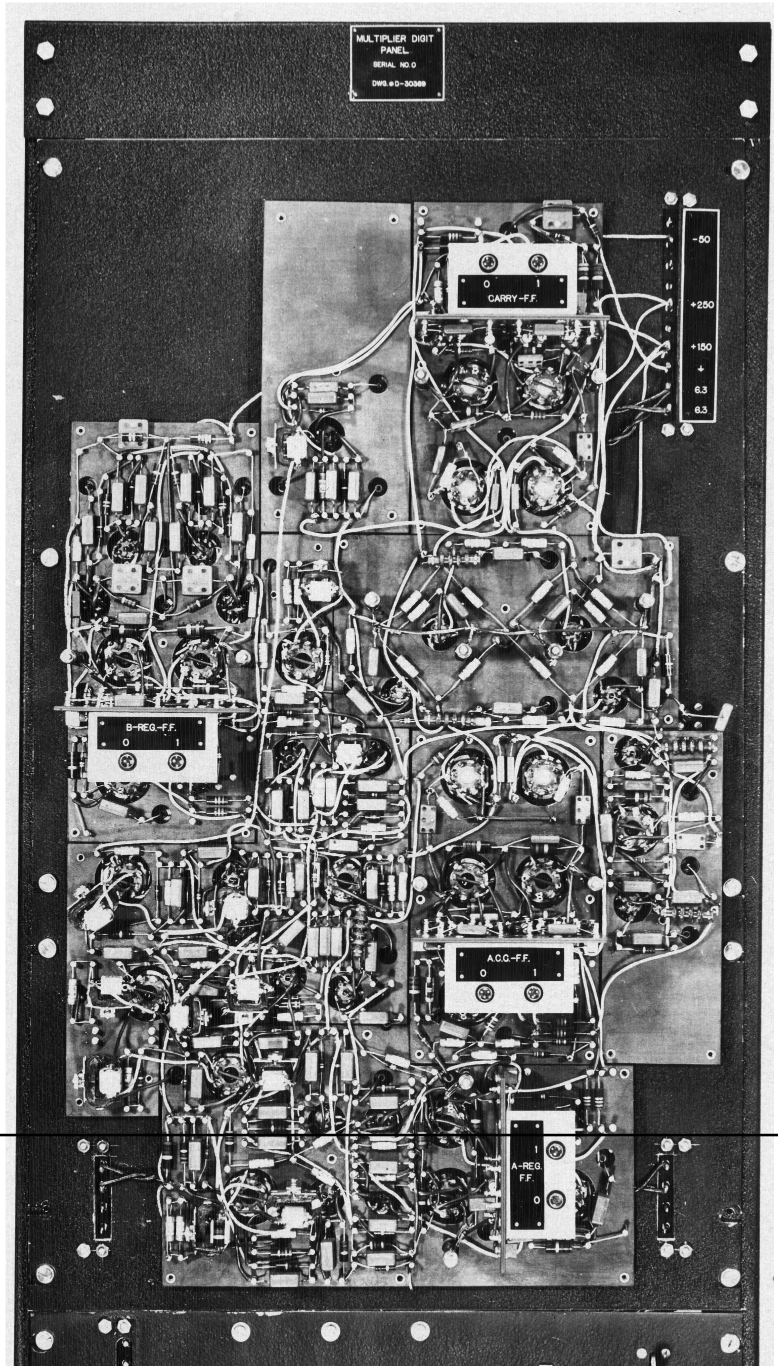
What in the world is this?



It's a 5-bit multiplier! Project Whirlwind 1946.



Norman H.Taylor
The Five-Digit Multiplier
R-134
Servomechanisms Laboratory,
MIT, December 3, 1948
(R-134 Vol.2 Schematics)



One bit of the accumulator:
all parts accessible at all times.

It is a carefully designed prototype of a 5-bit multiplier.

- Built as a proof of concept, to learn about technology, to check circuitry, and to understand errors.

- 2MHz clock; 5μsec multiplication time

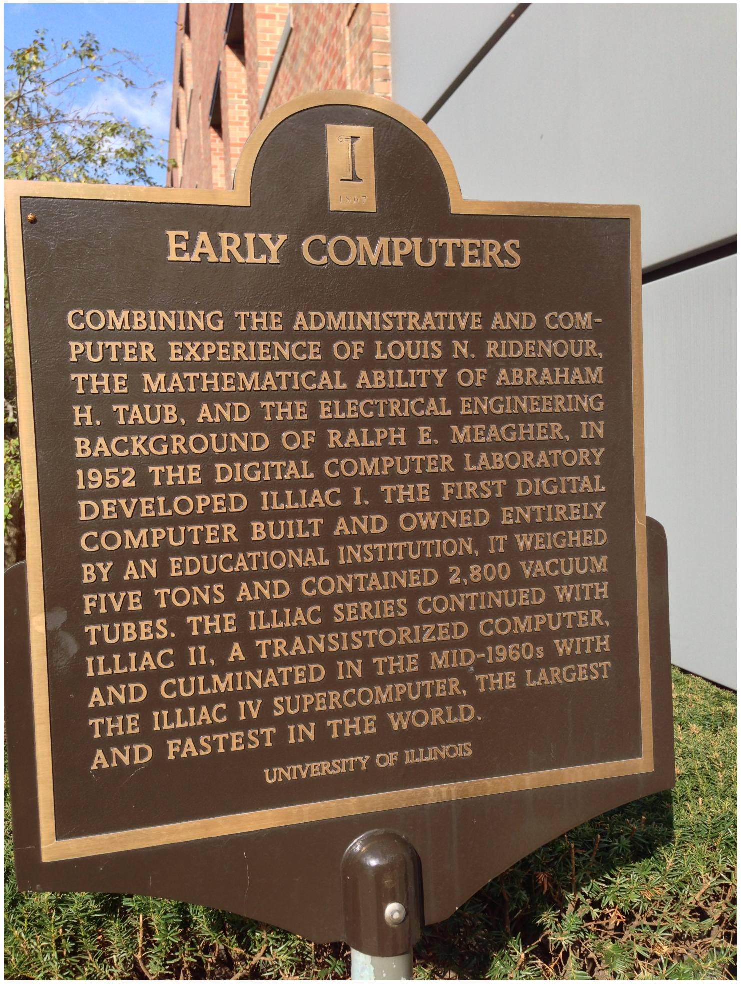
- 10 cycles: in odd cycles add, in even cycles perform ”shift and carry” operation (carry-save)

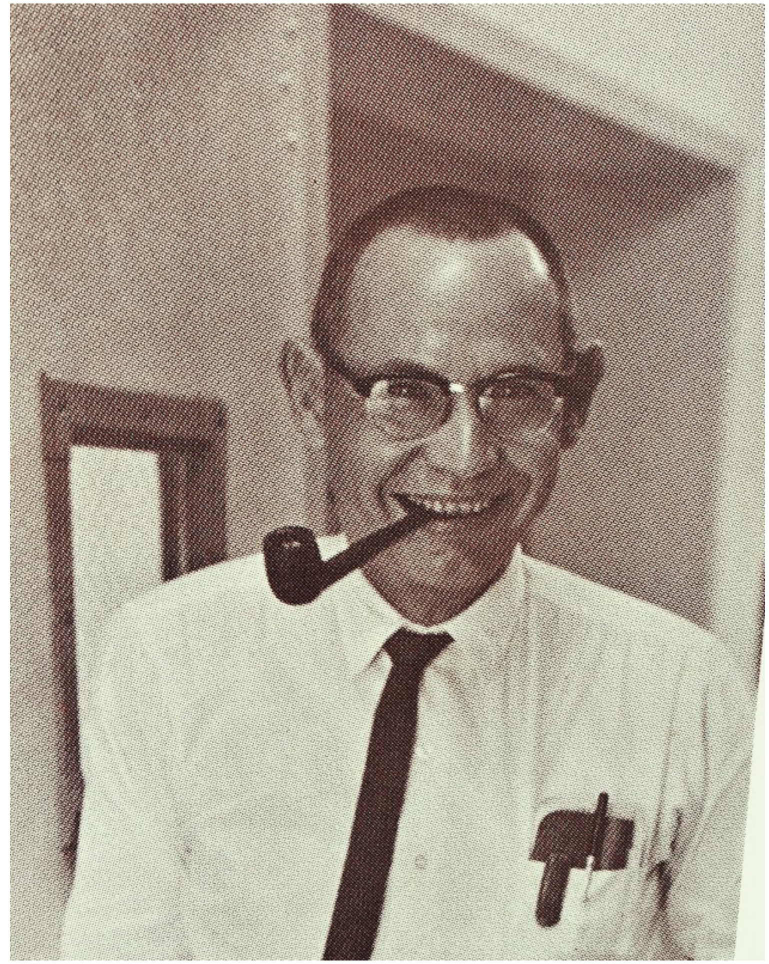
- probably the first electronic implementation of the carry-save concept

- 400 vacuum tubes

- a few weeks between errors

Illinois Days 1970-75 with Jim Robertson





HAL 9000 - the deceptive, clever, and sinister computer in Arthur C. Clarke's and Stanley Kubrick's 2001: A Space Odyssey, who "became operational at the HAL Plant in Urbana, Illinois"

Spreading Arithmetic Know-How

UNIVERSITY OF ILLINOIS

GRADUATE COLLEGE

DIGITAL COMPUTER LABORATORY

THEORY OF COMPUTER ARITHMETIC EMPLOYED IN THE DESIGN

OF THE NEW COMPUTER AT THE UNIVERSITY OF ILLINOIS

by

James E. Robertson

to be presented June 13-17, 1960

University of Michigan Engineering Summer Conference

"Theory of Computing Machine Design"

Early PhD Work in Arithmetic Guided by JER

G. Metze, *A Study of Parallel One's Complement Arithmetic Units with Separate Carry or Borrow Storage*, 1958.

A. Avizienis, *A Study of Redundant Number Representations for Parallel Digital Computers*, 1960.

J.O. Penhollow, *A Study of Arithmetic Recoding with Applications to Multiplication and Division*, 1962.

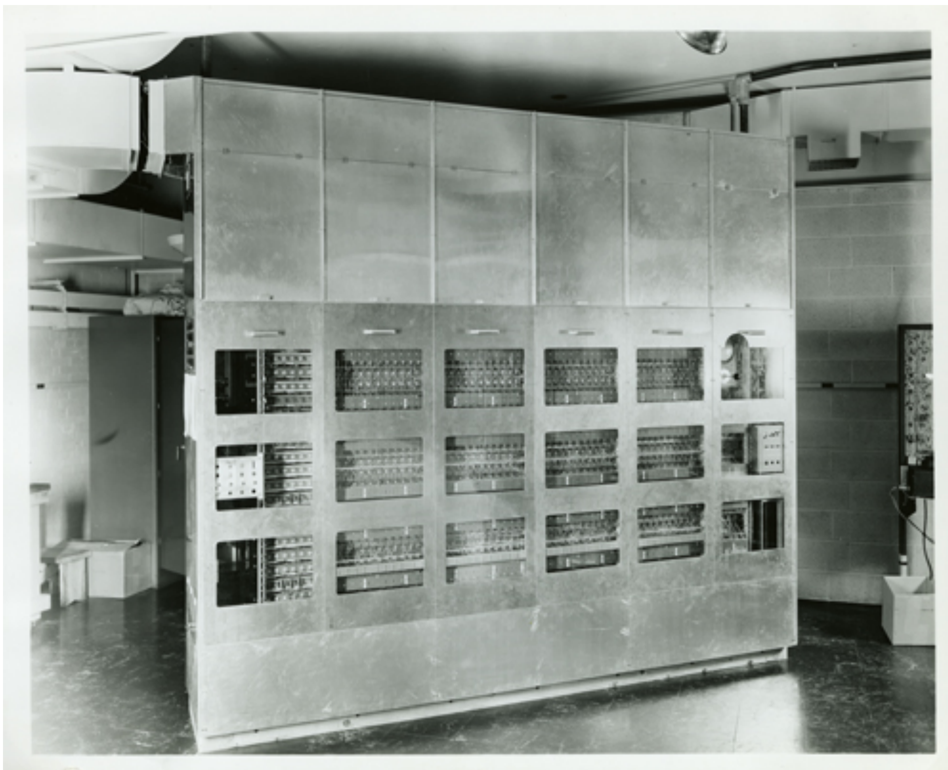
R. Shively, *Stationary Distributions of Partial Remainders in SRT Digital Division*, 1963.

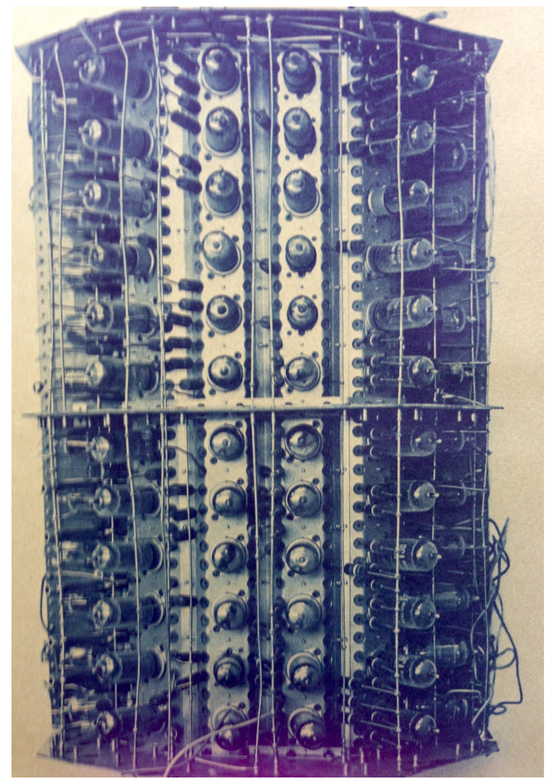
F. Rohatch, *A Study of Transformations Applicable to the Development of Limited Carry-Borrow Propagation Adders*, 1967.

Digital Computer Laboratory (DCL): home of ILLIACS and CS Dept.

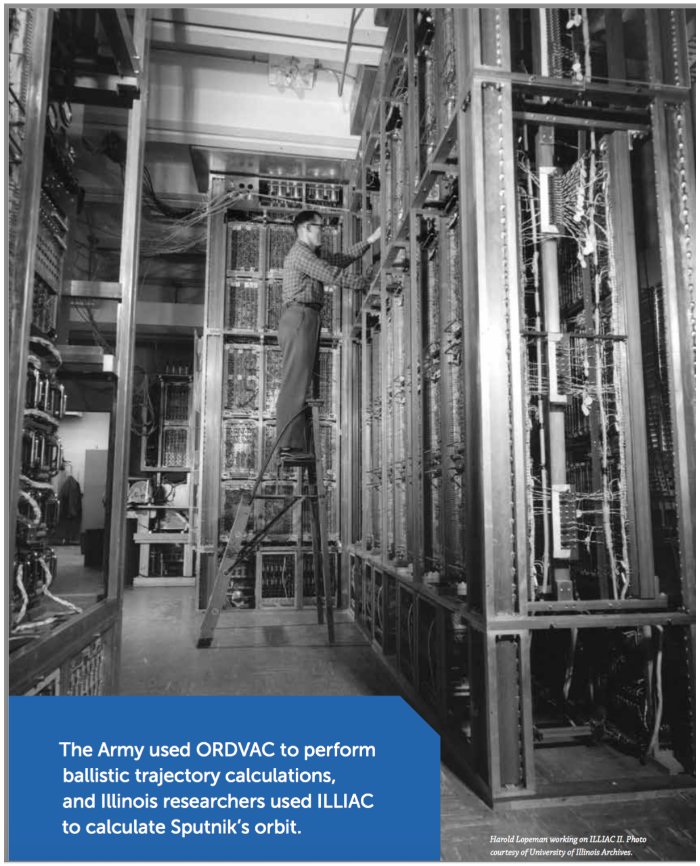
1949 - 1951: ORDVAC U.S.Army, 40-bit, add 92 μs, multiply 700 μs,

1952 - 1962: ILLIAC I (similar to ORDVAC design), 2,800 vacuum tubes, 5 tons





1957 - 1967: ILLIAC II: a breakthrough into a new generation of machines



On the Design of Very High-Speed Computer, DCL Report No. 80, Dec. 1957, widely read as a computer design Bible. Prepared by D.B. Gillies, R. E.Meagher, D.E. Muller, R.W Kay, J.P. Nash, J.E Robertson and A.H Taub. Initial 300 copies exhausted quickly.

- 100-200 times faster than ILLIAC I: add 0.3 μs; multiply 3.5-4 μs; division 7 - 20μs

Many ”firsts”:

- 15,400 transistors, 34,000 diodes and 42,000 resistors; 8K 52-bit words, 1.5 μs access time

- Separate carry storage (radix 4) and sum storage (radix 2).

- SRT radix 2 division unit designed by Jim Robertson.

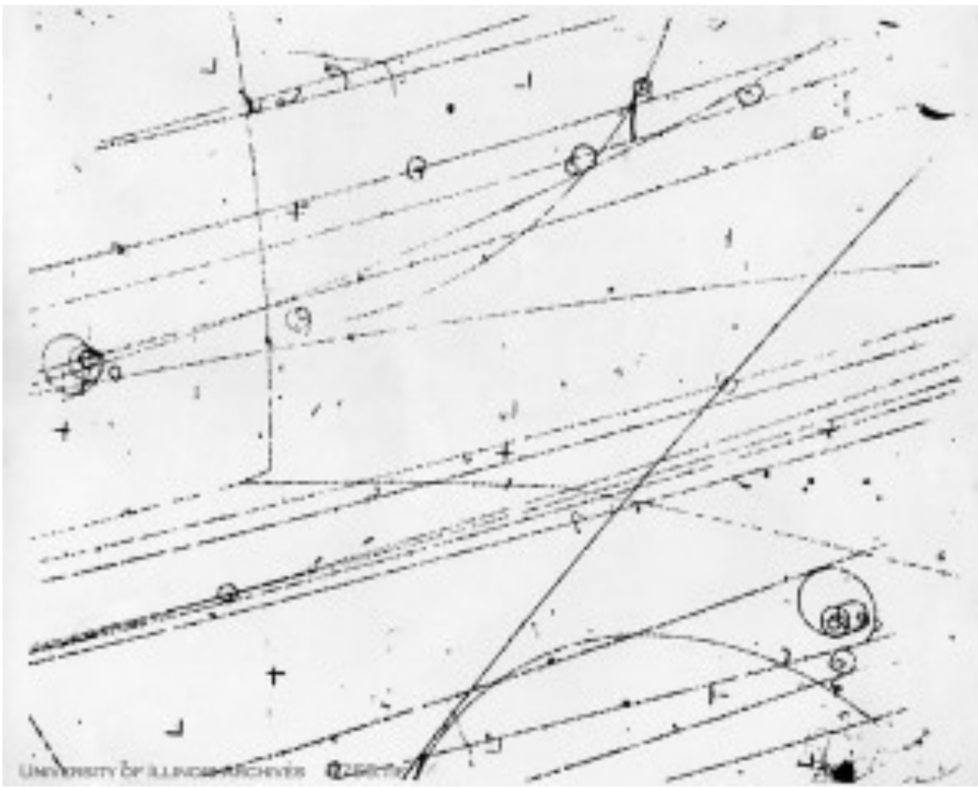
- IBM Stretch project borrowed many of its ideas from ILLIAC II

- One of the first pipelined computers: Advanced Control, Delayed Control, and Interplay stages (D. Gillies)

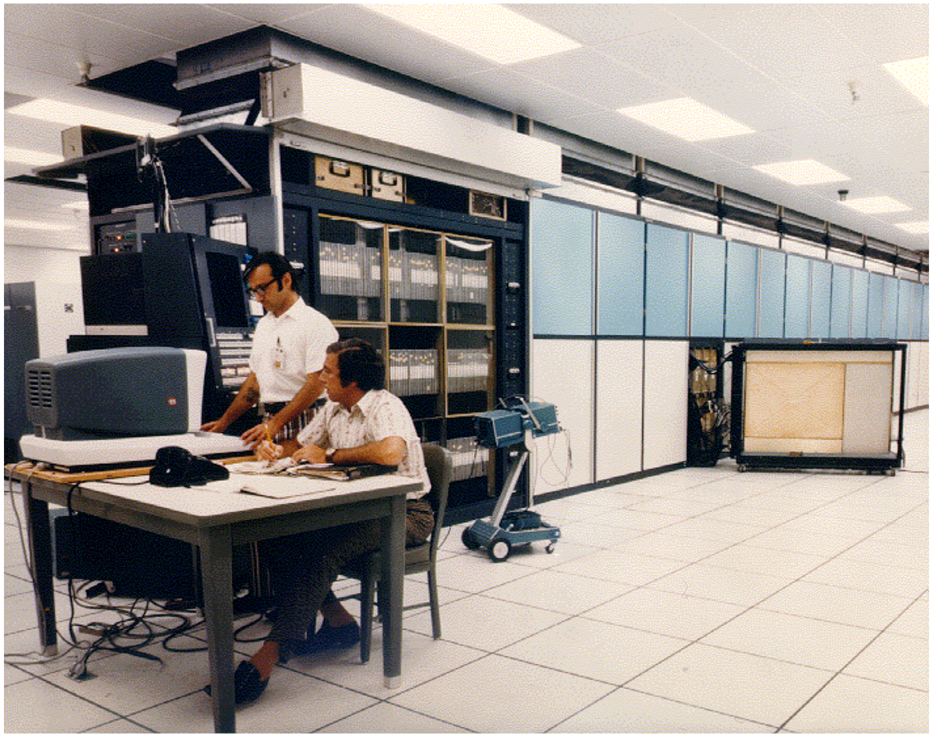
- The first computer to incorporate speed-independent circuitry in control unit (David E. Muller).

- In 1963 Donald Gillies discovers 3 new Mersenne primes (3000 digits) on Illiac-II in spare time. US Postal Service notices a bit later (1965).

1966 - 1970: ILLIAC III Pattern recognition - never finished. Good ideas, too early. 32 by 32 arithmetic elements. Anyway, some good arithmetic developed by Dan Atkins.



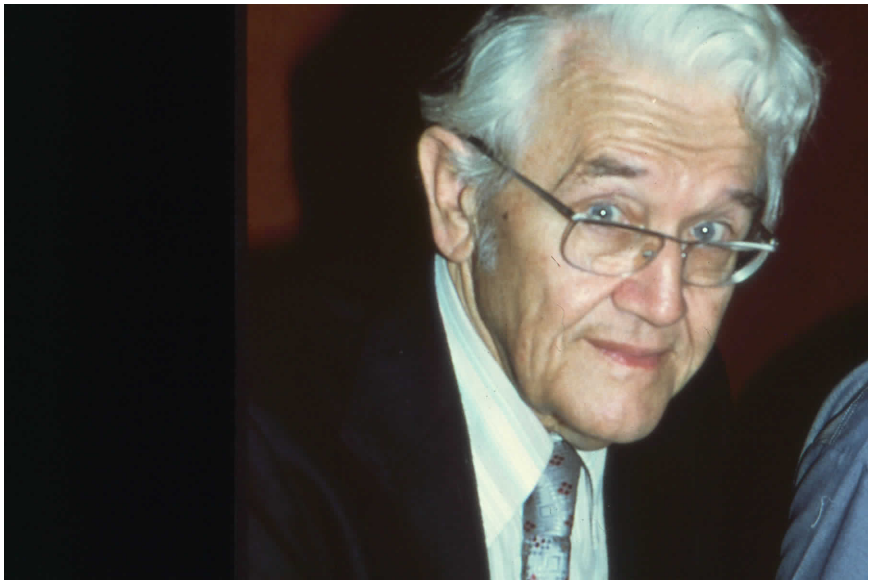
1965 - 1981: ILLIAC IV 64-processor SIMD supercomputer; 200 MFLOPs. Many ”firsts”, genesis of parallel compilers and parallel programming, interconnection networks, conflict-free access to parallel memories, ECL logic, 12-layer boards ...



Old DCL, still alive, inside a new building. CSD moved to Siebel Center and new heights.

Nostalgia Notes: ARITH-4, Santa Monica, 1978.

Antonin Svoboda, keynote speaker



From Prague, via Paris to MIT Radiation Lab - designing linkage computers for war efforts (Mark 56), back to Prague, proposing RNS arithmetic with Miroslav Valach and designing first fault-tolerant computers SAPO and EPOS, and, finally, to UCLA.

SAPO, a relay computer with stored-program capabilities, had three arithmetic units comparing the results to ensure correctness.

EPOS was a vacuum tube computer, utilizing RNS.

At UCLA he continued to work on arithmetic, logic design methods, and switching theory.

1950-1952 Svoboda helped design the M1 special-purpose relay computer for a 3D Fourier synthesis. Uses probably the first pipelined arithmetic unit to evaluate

$$\left[\sum_{n=1}^{N} \psi_n \sin\left(hx_n + ky_n + lz_n\right)\right]^2 + \left[\sum_{n=1}^{N} \psi_n \cos\left(hx_n + ky_n + lz_n\right)\right]^2$$

producing one result per one relay activation!

A man of many talents:

- a mathematician and a physicist

- an engineer

- authored *New Theory of Bridge* - a scientific theory of bidding strategies

- a pianist for the Prague Wind Quintet (Smetáček), a percussionist for the Czech Philharmonic Orchestra

- (Bohumil Martinu dedicated several pieces to him)

- an accomplished photographer and lens designer

- an expert APL programmer (his PRESTO, one of the first minimization programs, leading to IBM MIN, was written in APL)

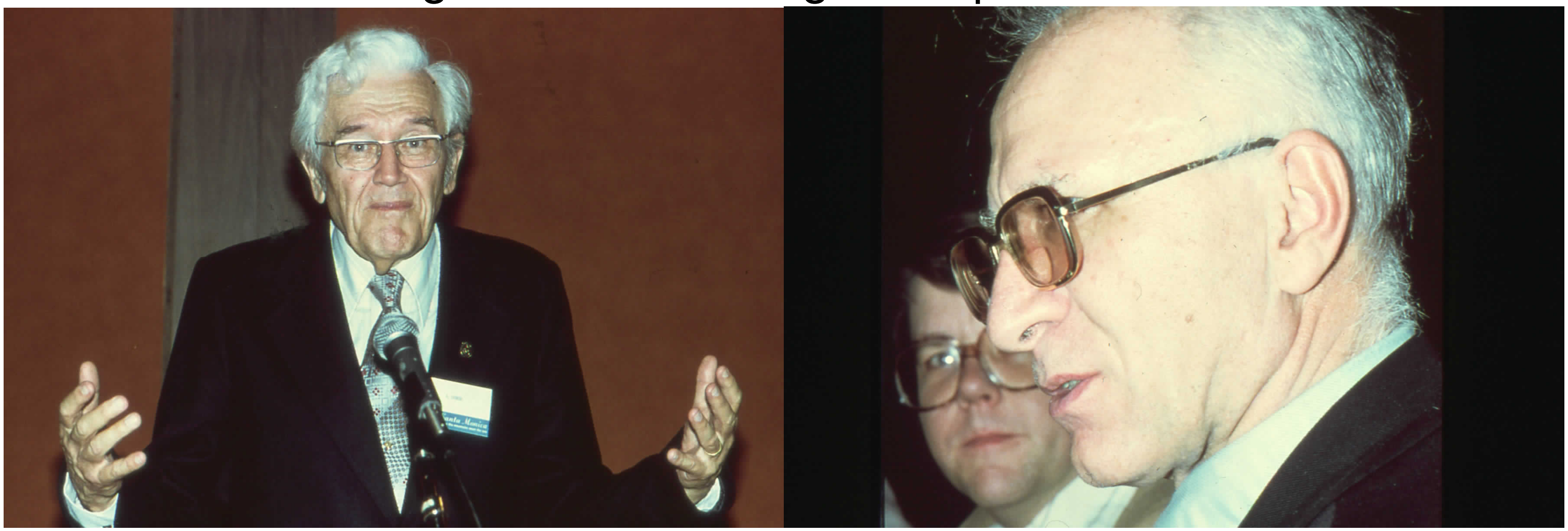
Best known for his pioneering work on the RNS theory and algorithms

Solved multiple-output minimization problem, disliked Karnaugh maps and loved Marquand charts (maps with binary labels); used punched cards to find prime implicants and minimal covers, ...,

Proposed a decimal division with simple selection, decimal arithmetic, arithmetic circuits with fault detection, ...

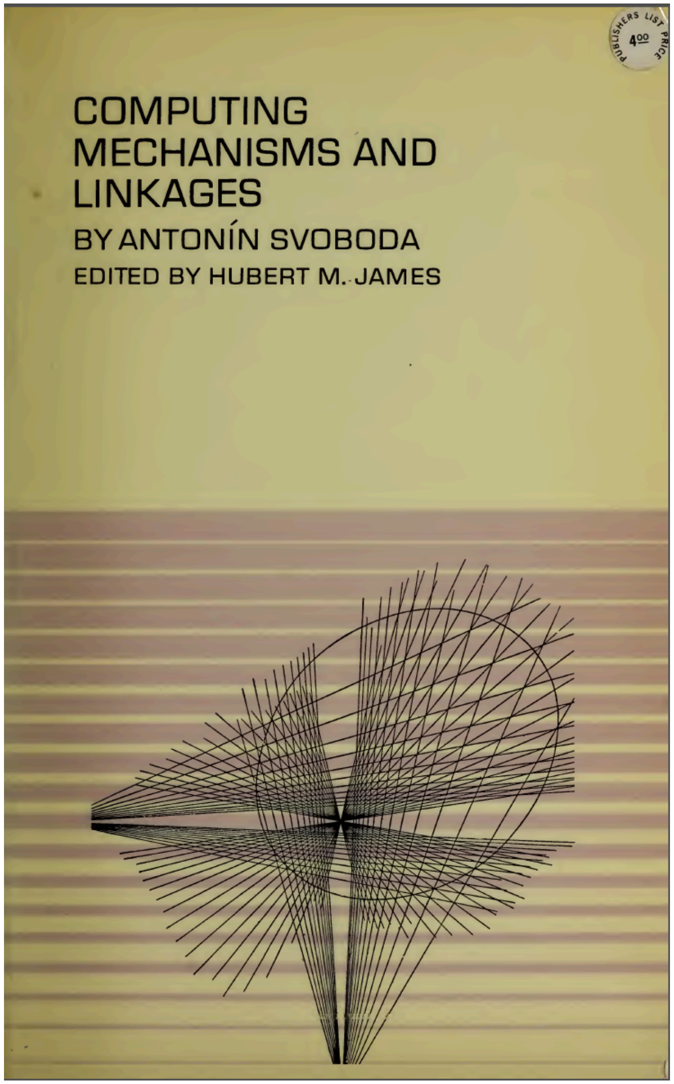
A witty, noble, exceptionally talented man with a charming ”akzent”, admired for his enthusiastic lectures

Dan Atkins and Luigi Dadda watching Toni speak at Arith-4



”I did not develop RNS primarily for speed. I also looked for a reliable design using unreliable components because I did not want to end up in Siberia. A truly modular design, inherent to RNS, was a key to increasing reliability.”

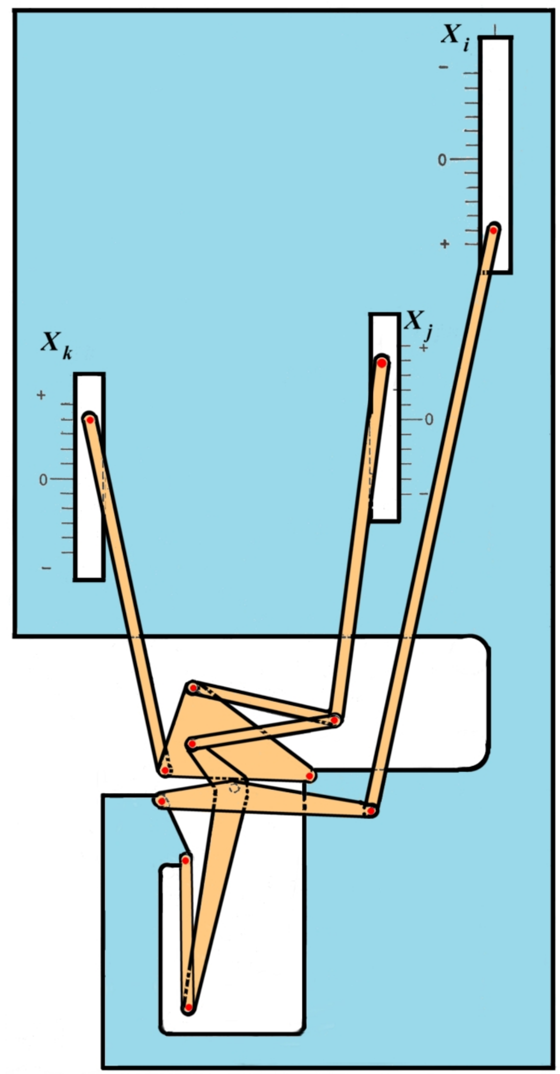
During the WWII, Svoboda developed methods for designing bar-linkage computers, for antiaircraft fire control. These are mechanical analogue computers, small, fast, reliable but difficult to design.



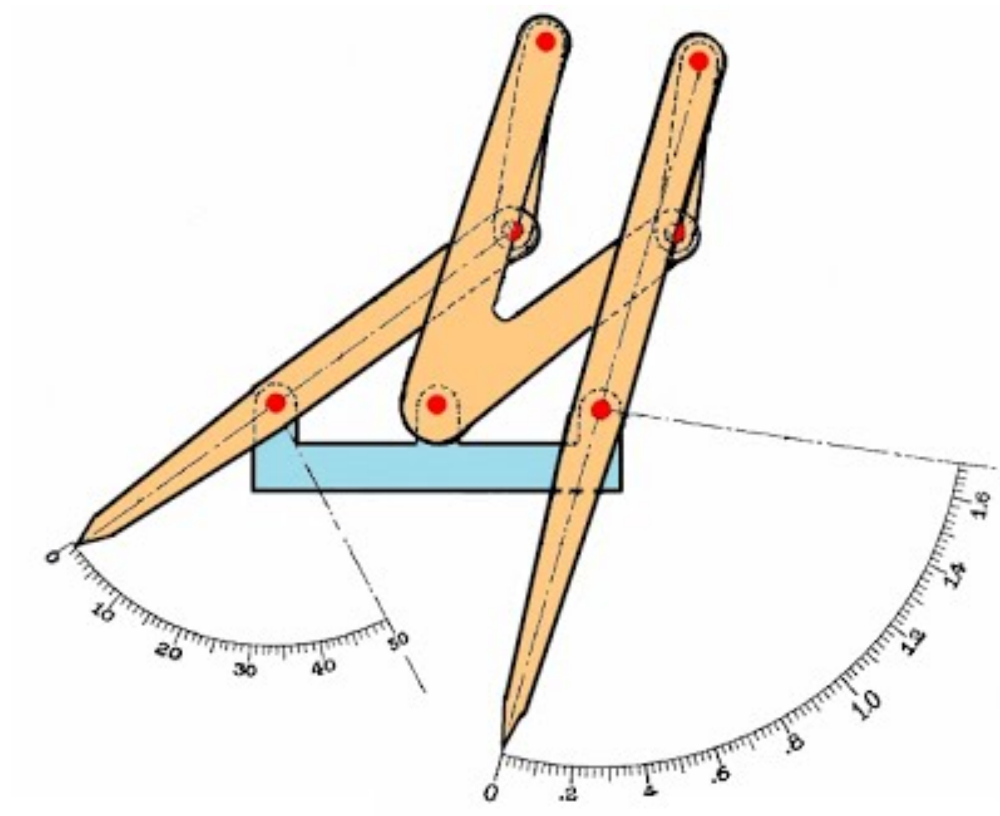
MIT Radiation Lab Linkage Computer of Mark 56: Toni on the right

Examples of elementary bar linkage "computers":

Multiplier $X_i = X_k \times X_j$



A double three-bar linkage is a logarithm generator with evenly spaced inputs $1 \leq X_i \leq 50$, and outputs with a maximum error of 0.003.

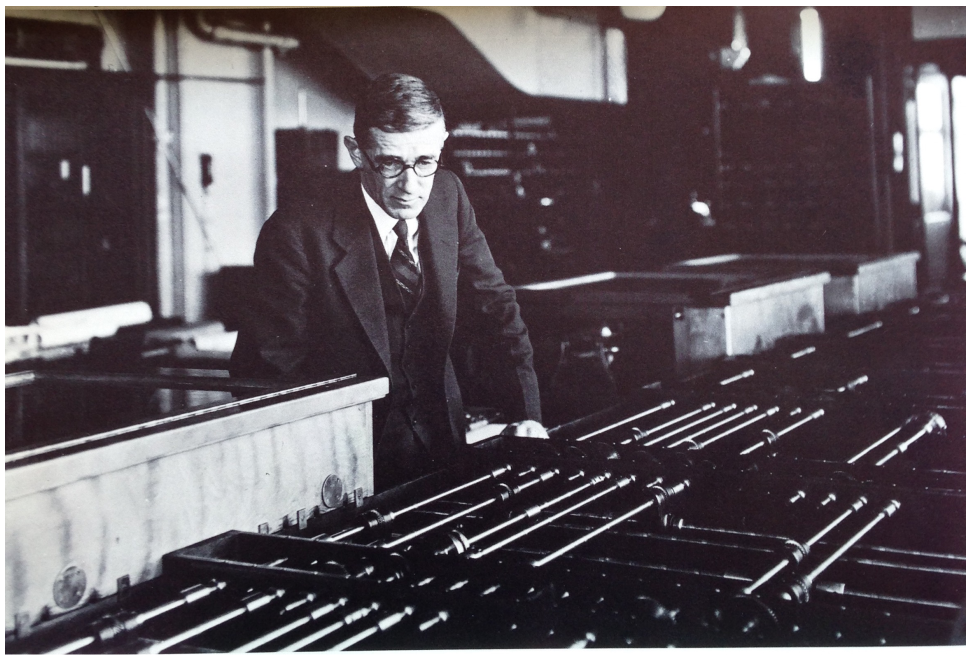


Arith-4, 1978, Santa Monica: with Jim Robertson and Robert Gregory

Arith-4, 1978, Santa Monica: with Algirdas Avižienis

36 years later, at the 2014 UCLA CSD Retreat

In 1947, UCLA installed a differential analyzer built for them by General Electric at a cost of $125,000, an electromechanical ”brain” of amazing capabilities. Needed regular lube and oil job. Created by Vannever Bush at MIT (of Memex fame, a knowledge repository with hyper-links.)



THANK YOU