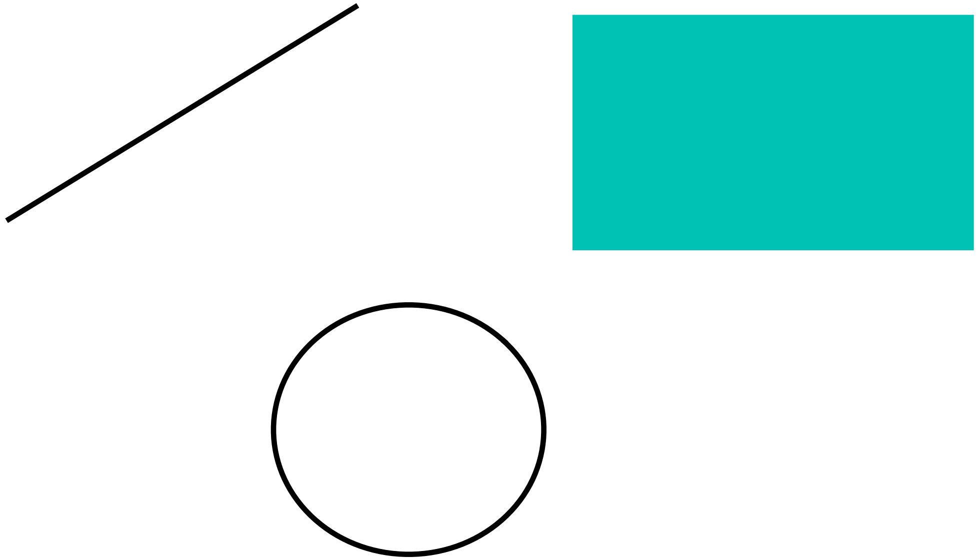
Primitives

Representations for
Lines and Curves



Representations for lines and Curves

Line (in 2D)

- Explicit

$$y = \frac{dy}{dx}(x - x_0) + y_0$$

- Implicit

$$F(x, y) = (x - x_0)dy - (y - y_0)dx$$

if	$F(x,y) = 0$	then	(x,y) is on line
	$F(x,y) > 0$		(x,y) is below line
	$F(x,y) < 0$		(x,y) is above line

- Parametric

$$x(t) = x_0 + t(x_1 - x_0)$$
$$y(t) = y_0 + t(y_1 - y_0)$$
$$t \in [0, 1]$$

$$P(t) = P_0 + t(P_1 - P_0), \text{ or}$$
$$P(t) = (1 - t)P_0 + tP_1$$

Circle

- Explicit

$$y = \pm\sqrt{r^2 - x^2}, \quad |\, x \,| \leq r$$

- Implicit

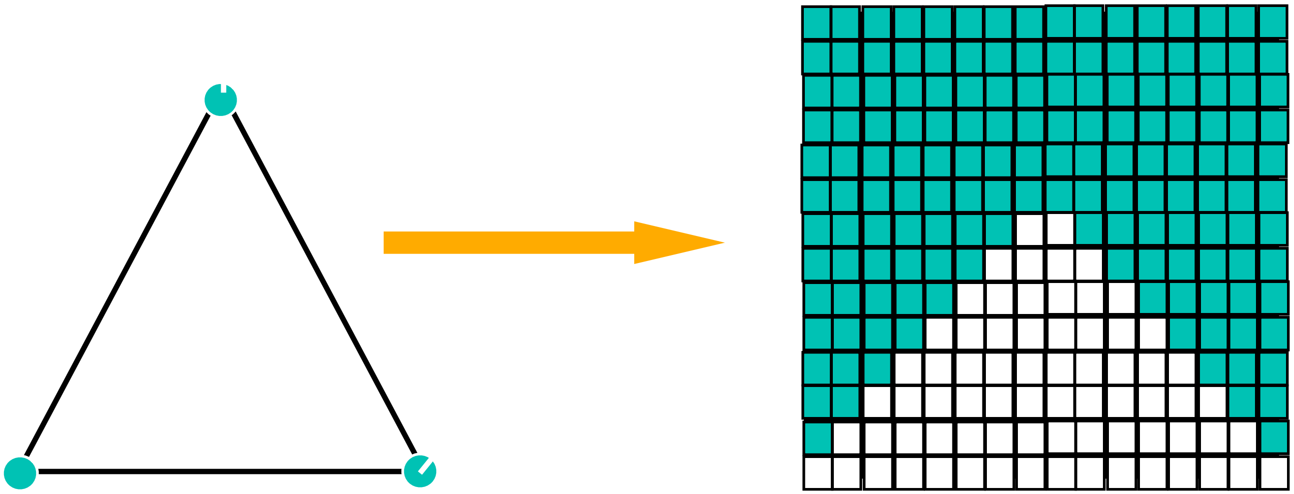
$$x^2 + y^2 = r^2$$
$$F(x, y) = x^2 + y^2 - r^2$$

$$\begin{array}{llll} \text{if} & F(x,y) = 0 & \text{then} & (x,y) \text{ is on circle} \\ & F(x,y) > 0 & & (x,y) \text{ is outside} \\ & F(x,y) < 0 & & (x,y) \text{ is inside} \end{array}$$

- Parametric

$$x(\theta) = r\cos(\theta)$$
$$y(\theta) = r\sin(\theta)$$
$$\theta \in [0, 2\pi]$$

Rasterization



Line rasterization

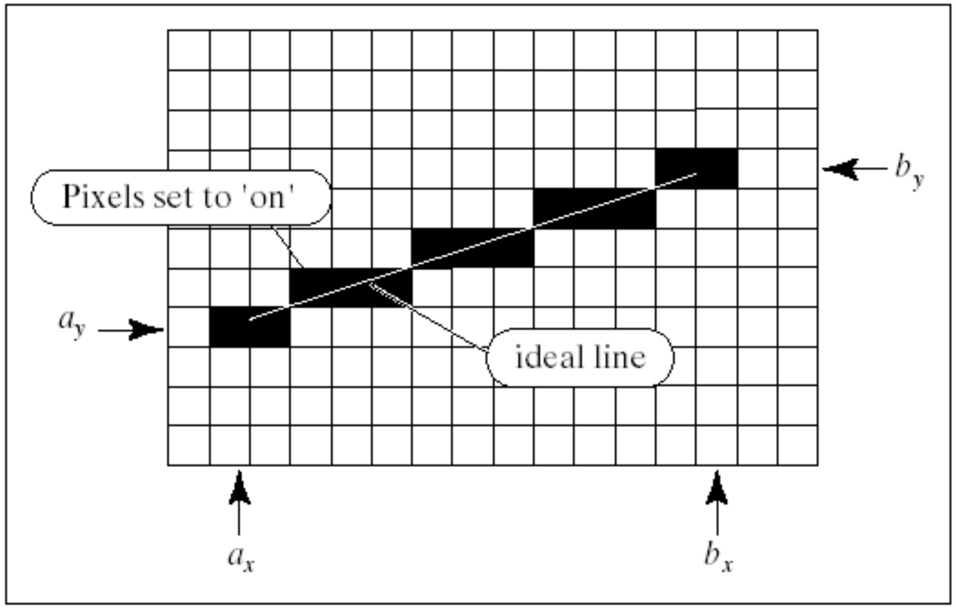


FIGURE 10.23 Drawing a straight-line-segment.

Line rasterization

Desired properties

- Straight
- Pass through end points
- Smooth
- Independent of end point order
- Uniform brightess
- Brightness independent of slope
- Efficient

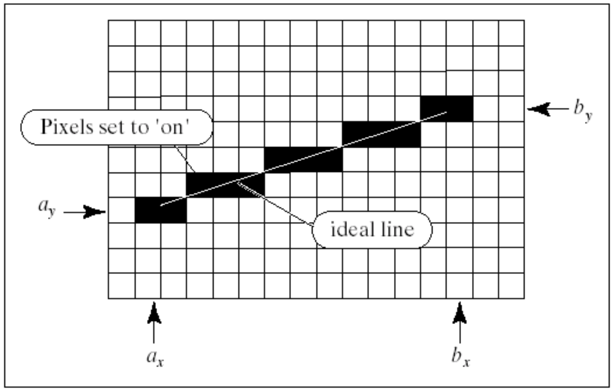


FIGURE 10.23 Drawing a straight-line-segment.

Straightforward Implementation

Straightforward Implementation

Line between two points

```
DrawLine(int x1,int y1,int x2,int y2)
    {
            float y;
            int x;
            for (x=x1; x<=x2; x++) {
                    y = y1 +  (x-x1)*(y2-y1)/(x2-x1)
                    SetPixel(x, Round(y) );
            }
    }
```

Better Implementation

How can we improve this algorithm?

```
DrawLine(int x1,int y1,int x2,int y2)
{
        float y;
        int x;
        for (x=x1; x<=x2; x++) {
                y = y1 +  (x-x1)*(y2-y1)/(x2-x1)
                SetPixel(x, Round(y) );
        }
}
```

Better Implementation

Better Implementation

```
DrawLine(int x1,int y1,int x2,int y2)
  {
          float y,m;
          int x;
          dx = x2-x1 ;
          dy = y2-y1 ;
          m = dy/ (float) dx ;
          for (x=x1; x<=x2; x++) {
               y = y1 + m*(x-x1) ;
               SetPixel(x, Round(y) );
          }
  }
```

Even Better Implementation

Even Better Implementation

```
DrawLine(int x1,int y1,int x2,int y2)
  {
                float y,m;
                int x;
                dx = x2-x1 ;
                dy = y2-y1 ;
                m = dy/ (float) dx ;
                y = y1 + 0.5 ;
                for (x=x1; x<=x2; x++) {
                        SetPixel(x, Floor(y) );
                        y = y + m ;
                }
  }
```

Midpoint algorithm (Bresenham)

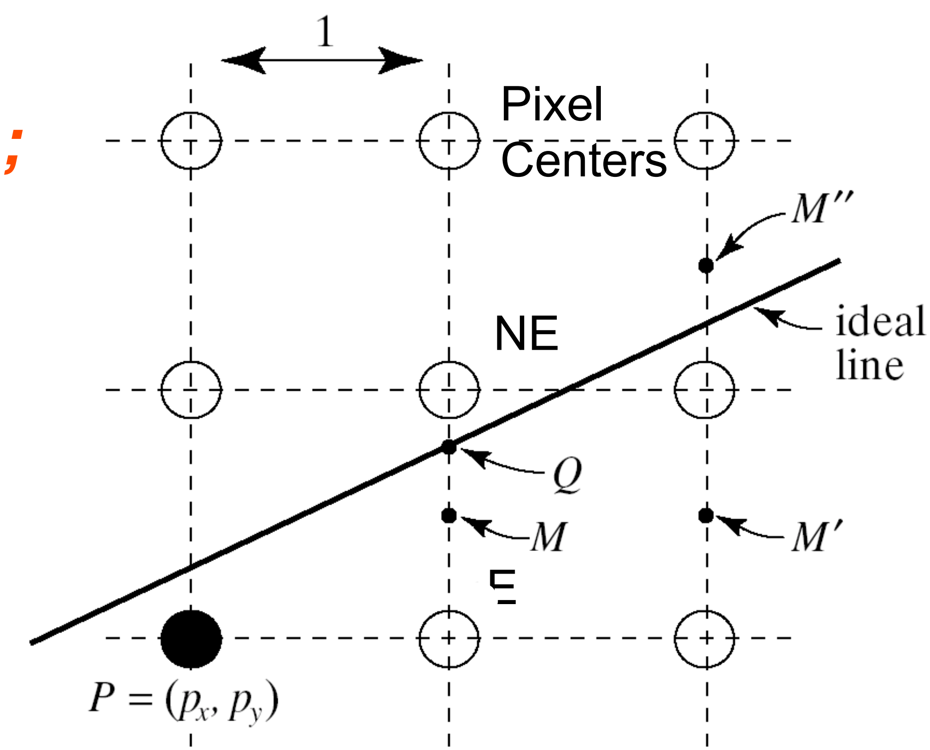
Line in the first quadrant (0<slope < 45 deg)

Implicit function:

F(x,y) = xdy - ydx + c,

dx,dy > 0 and dy/dx <= 1.0 ;

- Current choice P = (x,y).
- How do we chose next of P, P'= (x+1,y') ?



Midpoint algorithm (Bresenham)

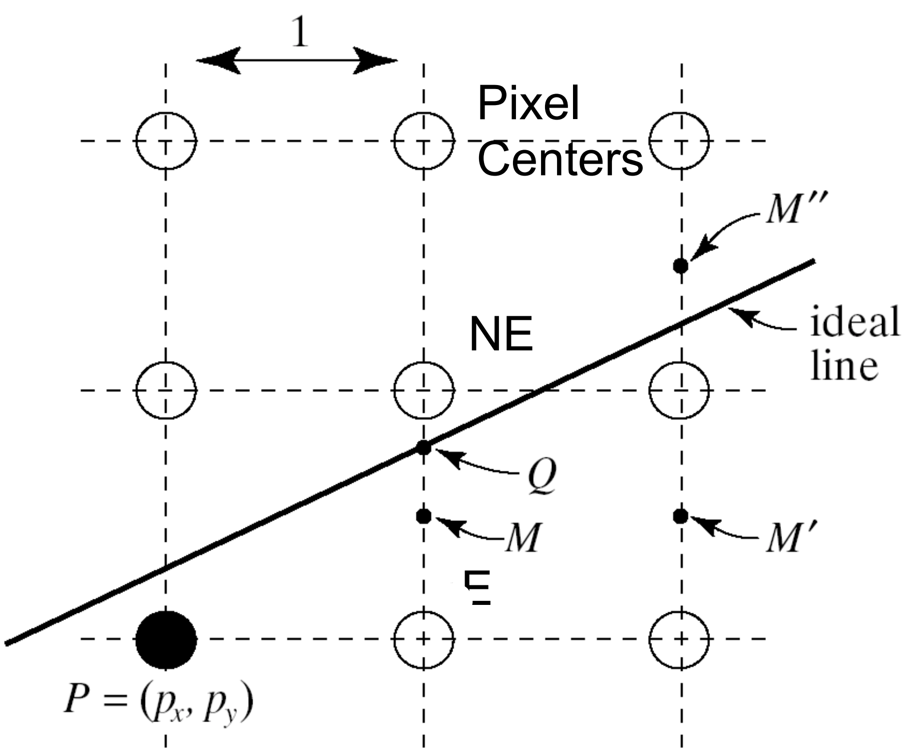
Line in the first quadrant (0<slope < 45 deg)

Implicit function:

F(x,y) = xdy - ydx + c,

dx,dy > 0 and dy/dx <= 1.0 ;

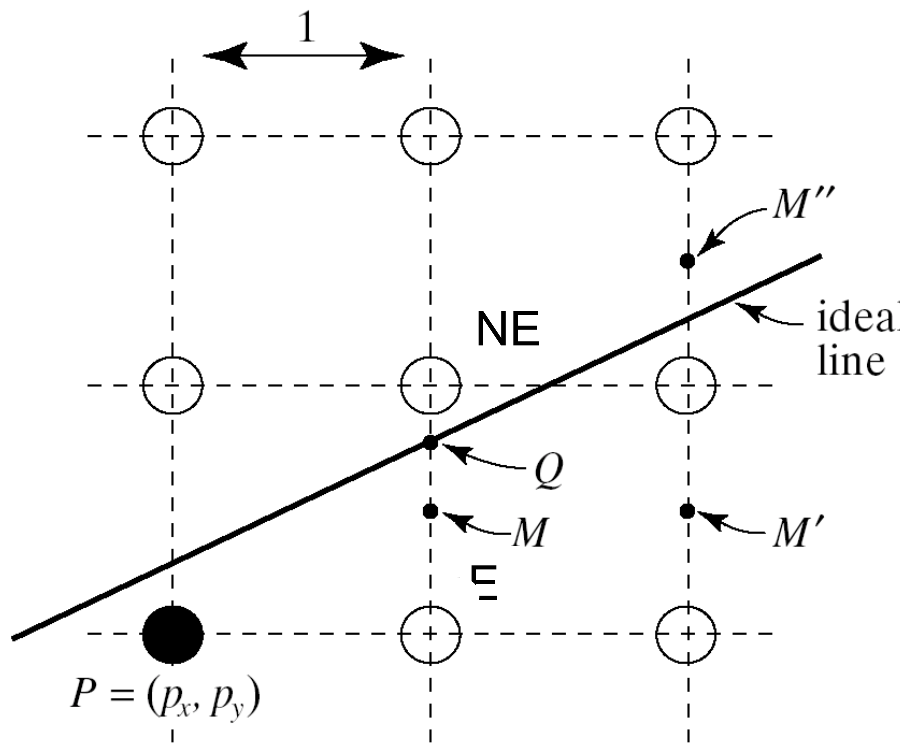
- Current choice P = (x,y).
- How do we chose next of P,
  P'= (x+1,y') ?
  If(F(M) = F(x+1,y+0.5) < 0)
  M above line so E
  else
  M below line so NE



Midpoint algorithm (Bresenham)

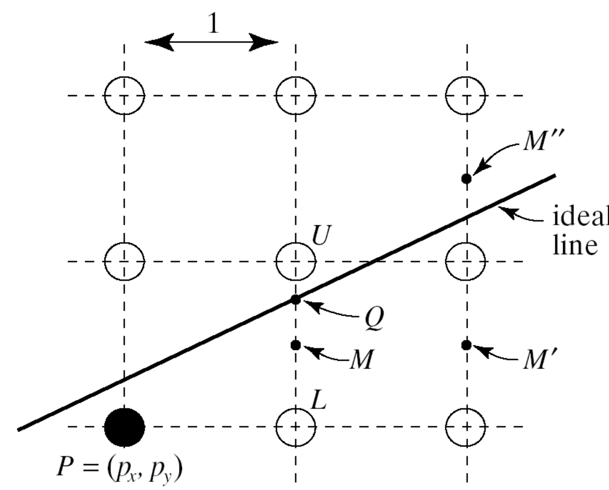
```
DrawLine(int x1, float y1, int x2, float y2, int color)
    {
        int x,y,dx,dy;

        y = Round(y1) ;
        for (x=x1; x<=x2; x++) {
            SetPixel(x, y );
            if (F(x+1,y+0.5)>0) {
                y = y + 1 ;
            }
        }
    }
```



Can we compute F in a smart way?

- We are at pixel (x,y) we evaluate F at M = (x+1,y+0.5) and E=(x+1,y) or NE=(x+1,y+1) accordingly.
  (Reminder: F(x,y) = xdy - ydx + c)



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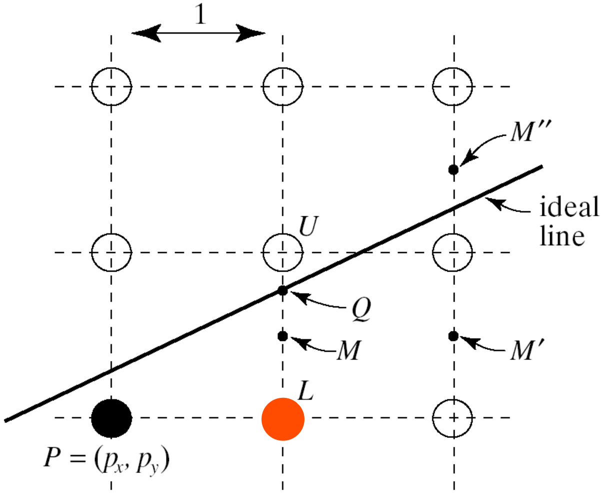
  (Reminder: F(x,y) = xdy - ydx + c)
- If we chose E for x+1 the next criteria will be at M':
  F(x+2,y+0.5) = [(x+1)dy +dy] - (y+0.5)*dx +c→
  F(x+2,y+0.5) = F(x+1,y+0.5) + dy →
  $F_E = F + dy = F + dF_E$



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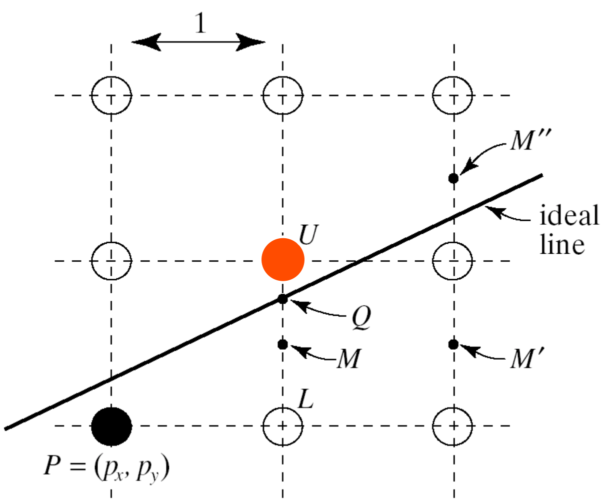
- We are at pixel (x,y) we evaluate F at M = (x+1,y+0.5) and E=(x+1,y) or NE=(x+1,y+1) accordingly.
  (Reminder: F(x,y) = xdy - ydx + c)
- If we chose E for x+1 the next criteria will be at M':
  F(x+2,y+0.5) = (x+1)dy +dy - (y+0.5)*dx +c→
  F(x+2,y+0.5) = F(x+1,y+0.5) + dy →
  F_E = F + dy
- If we chose NE then the next criteria will be at M'':
  F(x+2,y+1+0.5) =
  F(x+1,y+0.5) + dy – dx →
  F_{NE} = F + dy - dx



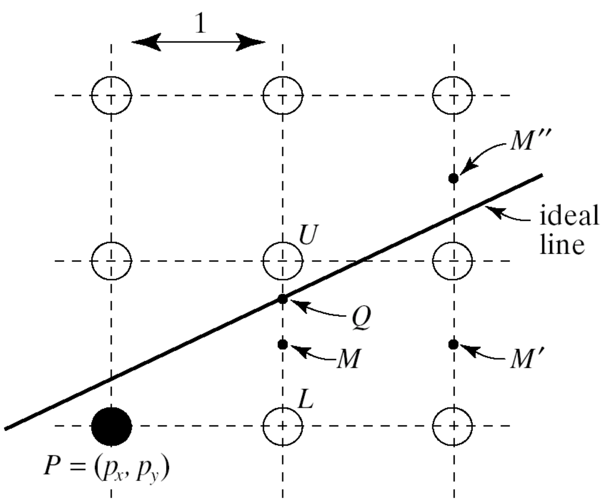
Can we compute F in a smart way?

- We are at pixel (x,y) we evaluate F at M = (x+1,y+0.5) and E=(x+1,y) or NE=(x+1,y+1) accordingly.
  (Reminder: F(x,y) = xdy - ydx + c)
- If we chose E for x+1 the next criteria will be at M':

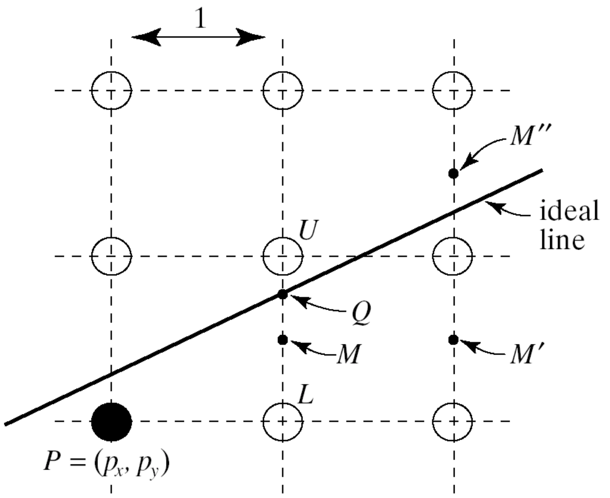
$F_E = F + dy$

- If we chose NE then the next criteria will be at M'':

$F_{NE} = F + dy - dx$



Criterion update



Update

$F_E = F + dy = F + dF_E$

$F_{NE} = F + dy - dx = F + dF_{NE}$

Starting value?

Line equation: $F(x,y) = xdy - ydx + c$

Assume line starts at pixel (x_0,y_0)

$$F_{start} = F(x_0+1, y_0+0.5) = (x_0+1)dy - (y_0+0.5)dx + c =$$

$$= (x_0 dy - y_0 dx + c) + dy - 0.5dx = F(x_0,y_0) + dy - 0.5dx.$$

(x_0,y_0) belongs on the line so: $F(x_0,y_0) = 0$

Therefore:

$$F_{start} = dy - 0.5dx$$

Criterion update (Integer version)

Update

F_{start} = dy –0.5dx

F_E = F + dy = F + dF_E

F_{NE} = F + dy - dx = F + dF_{NE}

Everything is integer except F_{start}.

Multiply by 2 → F_{start} = 2dy – dx

dF_E = 2dy

dF_{NE} = 2(dy-dx)

Midpoint algorithm

```
DrawLine(int x1, float y1, int x2, float y2, int color)
    {
        int x,y,dx,dy,dE, dNE;
        dx = x2-x1 ;
        dy = y2-y1 ;
        d = 2*dy-dx ; // initialize d
        dE = 2*dy ;
        dNE = 2*(dy-dx) ;
        y = Round(y1) ;
        for (x=x1; x<=x2; x++) {
            SetPixel(x, y, color );
            if (d>0) {      // chose NE
                d = d + dNE ;
                y = y + 1 ;
            } else {        // chose E
                d = d + dE ;
            }
        }
    }
```

Incremental algorithms for polynomials

$$
\begin{aligned}
F(x) &= a_n x^n + a_{n-1} x^{n-1} \ldots + a_1 x + a_0, a_n \neq 0 \\
F(x+d) &= a_n (x+d)^n + \boxed{a_{n-1}(x+d)^{n-1} \ldots + a_1 (x+d) + a_0} = \\
&= a_n (x+d)^n + P^{n-1}(x) \\
&= a_n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} d^k + P^{n-1}(x) \\
&= a_n \sum_{k} \left(\frac{n}{k!(n-k)!} \right) x^{n-k} d^k + P^{n-1}(x) \\
&= a_n x^n + \sum_{k=1}^{n} \left(\frac{n}{k!(n-k)!} \right) x^{n-k} d^k + P^{n-1}(x) \\
&= a_n x^n + R^{n-1}(x) + P^{n-1}(x)
\end{aligned}
$$

N-order differences

$$\begin{aligned} F(x) &= a_n x^n + a_{n-1} x^{n-1} \ldots + a_1 x + a_0, a_n \neq 0 \\ F(x+d) &= a_n x^n + R^{n-1}(x) + P^{n-1}(x) \end{aligned}$$

First order

$$\Delta F = F(x+d) - F(x) = R^{n-1}(x) + P^{n-1}(x) = G_1^{n-1}(x)$$

N-order

$$\begin{aligned} \Delta^2 F(x) &= \Delta F(x+d) - \Delta F(x) = G_2^{n-2}(x) \\ &\vdots \\ \Delta^n F(x) &= \Delta^{n-1} F(x+d) - \Delta^{n-1} F(x) = G_n^0 = c \end{aligned}$$

N-order difference update

$$\begin{aligned} F(x) &= a_n x^n + a_{n-1} x^{n-1} \ldots + a_1 x + a_0, a_n \neq 0 \\ F(x+d) &= a_n x^n + R^{n-1}(x) + P^{n-1}(x) \end{aligned}$$

$$\begin{aligned} F(x+d) &= F(x) + \Delta F(x) \\ \Delta F(x+d) &= \Delta F(x) + \Delta^2 F(x) \\ &\vdots \\ \Delta^{n-1} F(x+d) &= \Delta^{n-1} F(x) + \Delta^n F(x) \\ \Delta^n F(x+d) &= c \end{aligned}$$

We need n initial conditions to initialize the differences.

Example: y = x^2

$$
\begin{aligned}
y(x+d) &= x^2 + 2xd + d^2 = y(x) + 2xd + d^2 \\
\rightarrow \quad & y(x+d) = y(x) + \Delta y(x) \\
& where \quad \Delta y(x) = 2xd + d^2 \\
\Delta y(x+d) &= 2(x+d)d + d^2 = \Delta y(x) + 2d^2 \\
\rightarrow \quad & \Delta y(x+d) = \Delta y(x) + \Delta^2 y(x) \\
& where \quad \Delta^2 y(x) = 2d^2
\end{aligned}
$$

The incremental algorithm to compute y = x^2

```
computePar(int d)
  {
            float y = 0 ;
            int x = 0 ;
            DY = d^2 ; // at x = 0
            DDY = 2*d^2 ;
            for( x = 0 ; x < X_MAX ; x++ ) {
                  printf(“d, %f\”, x,y) ;
                  y = y + DY ;
                  DY = DY + DDY ;
  }
```

Polygon

Collection of points connected with lines

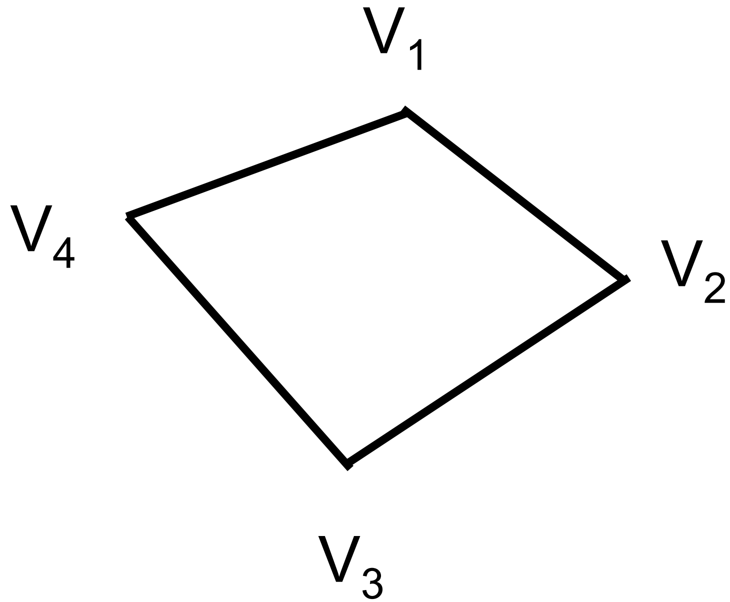
- Vertices: v1,v2,v3,v4
- Edges:

$e_1 = v_1v_2$

$e_2 = v_2v_3$

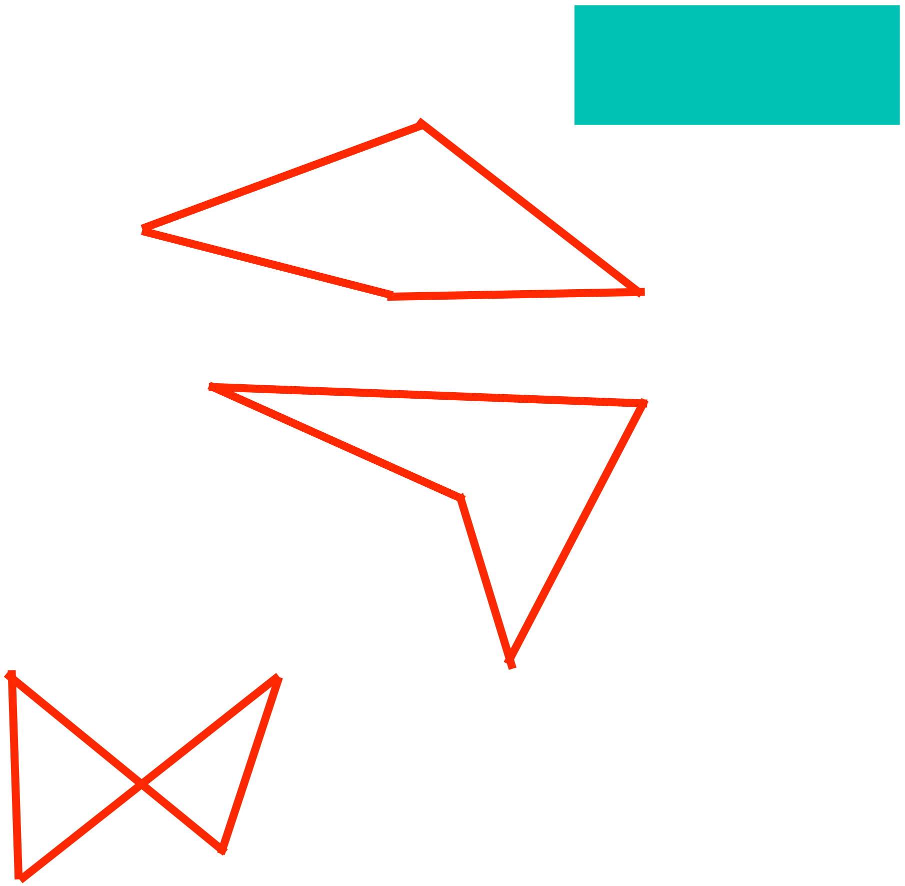
$e_3 = v_3v_4$

$e_4 = v_4v_1$



Polygons

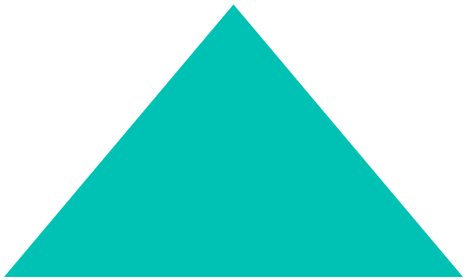
- Open / closed
- Planar / non-planar
- Filled / wireframe
- Convex / concave
- Simple / non-simple



Triangles

The most common primitive

- Convex
- Planar
- Simple



Background

Plane equations

Implicit

$F(x,y,z) = Ax + By + Cz + D = \mathbf{N} \bullet P + D$

Points on Plane $F(x,y,z) = 0$

Parametric

$Plane(s,t) = P_0 + s(P_1 - P_0) + t(P_2 - P_0)$

P_0, P_1, P_2 not colinear

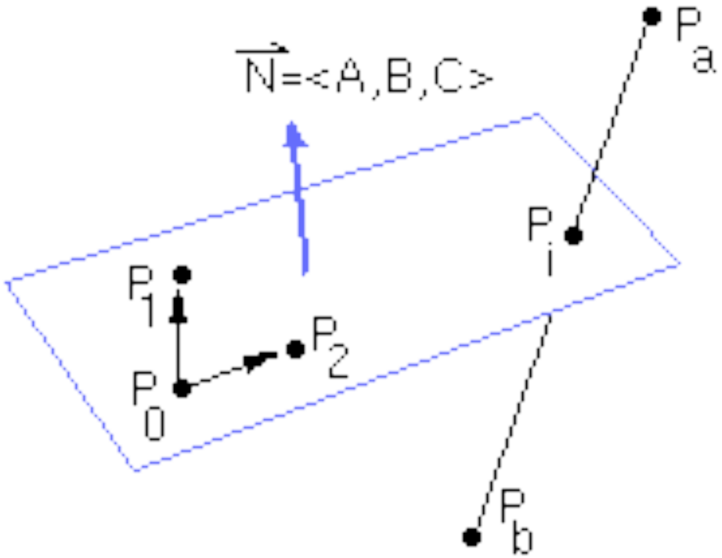
or

$Plane(s,t) = (1 - s - t)P_0 + sP_1 + tP_2$

$Plane(s,t) = P_0 + sV_1 + tV_2$ where V_1, V_2 basis vectors

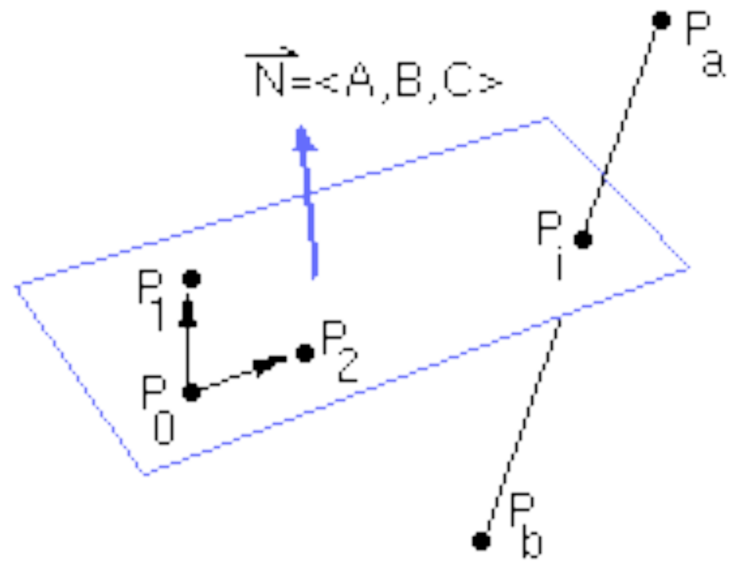
Explicit

$z = -(A/C)x - (B/C)y - D/C, \; C \neq 0$



Point normal form

Plane equation



$$F(x,y,z) = Ax + By + Cz + D = \mathbf{N} \bullet P + D$$

Points on Plane $F(x,y,z) = 0$

Observation : Let's take an arbitrary vector u that lies on the plane which can be defined by two points e.g. P1, P2 on the plane.

$$\mathbf{u} = P2 - P1$$

$$\left.\begin{array}{l} \mathbf{N} \bullet P1 + D = 0 \\ \mathbf{N} \bullet P2 + D = 0 \end{array}\right\} \Rightarrow \mathbf{N} \bullet (P2 - P1) = 0 \Rightarrow \mathbf{N} \bullet \mathbf{u} = 0 \Rightarrow \boxed{\mathbf{N} \perp \mathbf{u}}$$

Computing point normal form from 3 Points

$F(x, y, z) = Ax + By + Cz + D = \mathbf{N} \bullet P + D$

Points on Plane $F(x, y, z) = 0$

First way :

$\mathbf{N} \bullet P0 + D = 0$

$\mathbf{N} \bullet P1 + D = 0$

$\mathbf{N} \bullet P2 + D = 0$

$|\mathbf{N}| = 1$ (arbitrary choice)

Second way :

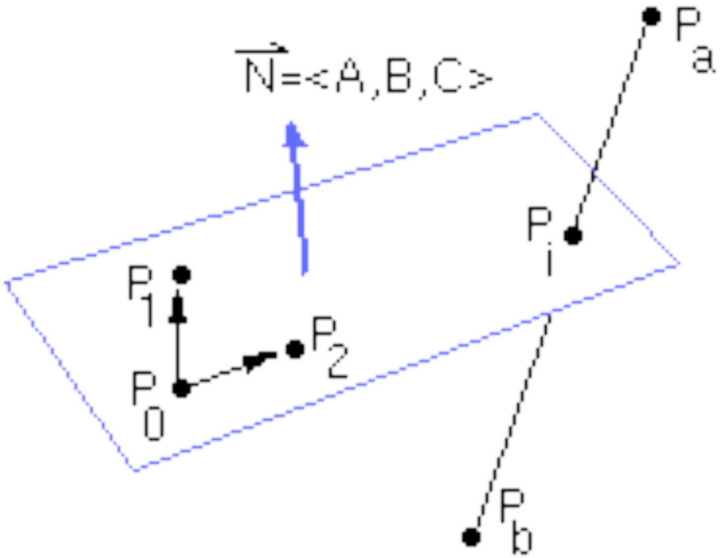
$\mathbf{N}$ is normal to F

Let's find a normal vector :

$\mathbf{N} = (P1 - P0) \times (P2 - P0)$

Compute D :

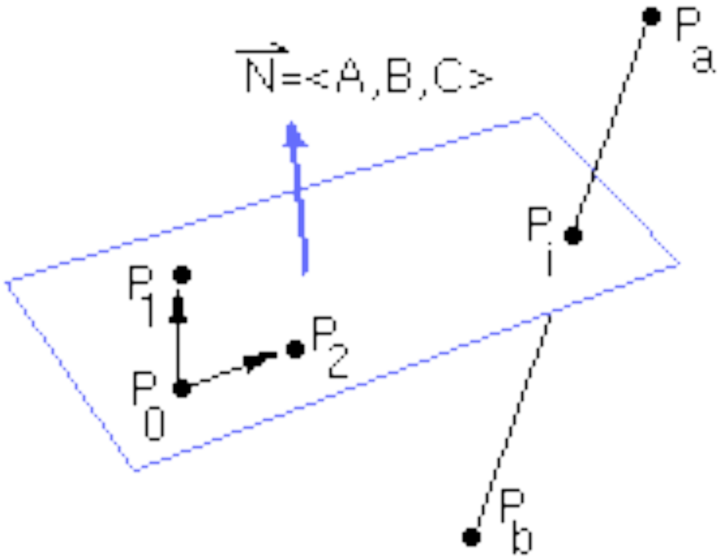
$D = -\mathbf{N} \bullet P0$



Intersection of lines and planes

Plane: Pl(P) = N*P +D = 0

Line: Pa+t(Pb-Pa), t in R



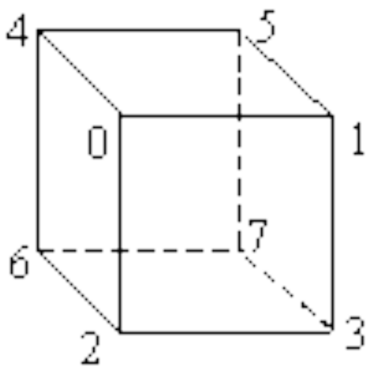
$$\vec{N} \bullet (P_a + t\,(P_b - P_a)) + D = 0$$

$$t = \frac{-D - \vec{N} \bullet P_a}{\vec{N} \bullet P_b - \vec{N} \bullet P_a} = \frac{-F(P_a)}{F(P_b) - F(P_a)}$$

Polygonal models/ data structures

[Hill: *p. 287-291. Foley & van Dam: p. 471-477*]

Indexed face set



faces

#	vertex list
0	0,2,3,1
1	1,3,7,5
2	5,7,6,4
3	4,6,2,0
4	4,0,1,5
5	2,6,7,3

vertex list

#	x,y,z
0	0,1,1
1	1,1,1
2	0,0,1
3	1,0,1
4	0,1,0
5	1,1,0
6	0,0,0
7	1,0,0

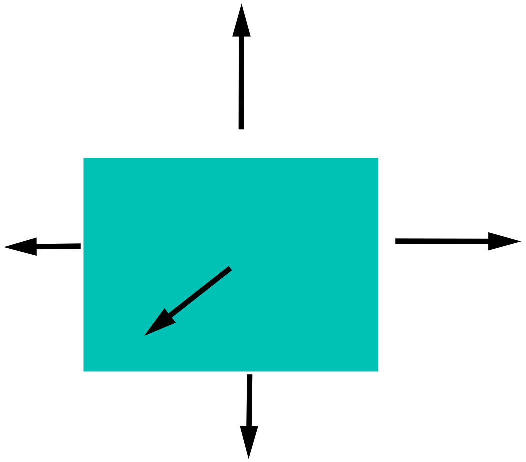
Polygon attributes

Per vertex or per face

- Normal
- Color

Per vertex

- Texture coordinates



Computing the normal of a polygon

One way:

$N = (V_{n-1}-V_0)x(V_1-V_0)$

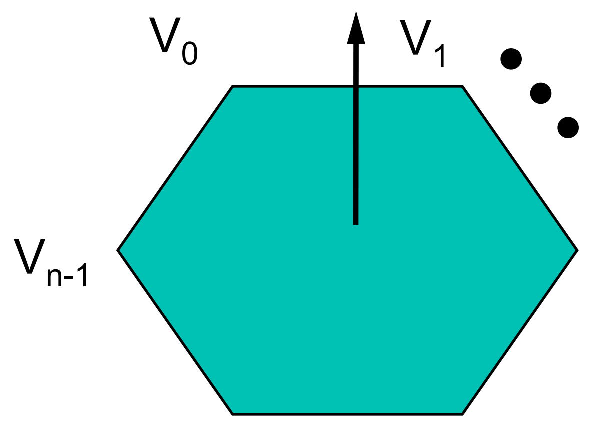
Newell's method (page 292)

$$N_x = \sum_{i=0}^{n-1}(y_i - y_{next(i)})(z_i + z_{next(i)})$$

$$N_y = \sum_{i=0}^{n-1}(z_i - z_{next(i)})(x_i + x_{next(i)})$$

$$N_z = \sum_{i=0}^{n-1}(x_i - x_{next(i)})(y_i + y_{next(i)})$$

$$where \quad next(j) = (j+1) \ mod \ n$$



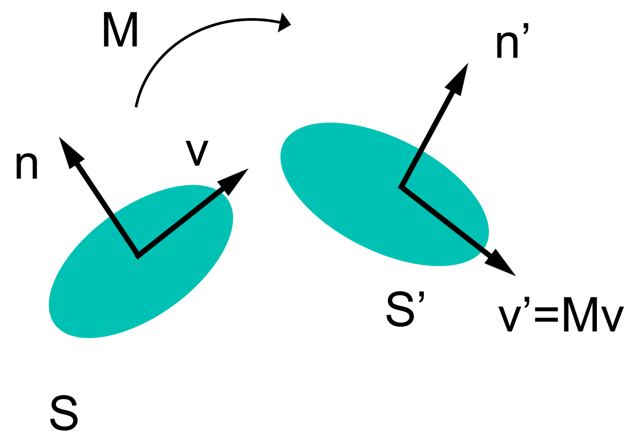
Transforming Normals

Normal vectors are transformed along with vertices and polygons.

- How do you transform a normal ?
- What about unit magnitude ?

Deriving transformation of normal

$$\mathbf{n} = (n_x, n_y, n_z, 0)^T \quad normal \quad to \quad S$$
$$\mathbf{v} = (v_x, v_y, v_z, 0)^T \quad tangent \quad to \quad S$$
$$S' = MS, \quad what \quad is \quad \mathbf{n}'?$$



$$\mathbf{n} \cdot \mathbf{v} = \mathbf{n}^T \mathbf{v} = 0$$
$$\mathbf{n}^T \mathbf{v} = 0 \rightarrow \mathbf{n}^T I \mathbf{v} = 0 \rightarrow \mathbf{n}^T (M^{-1} M) \mathbf{v} = 0$$
$$\rightarrow (\mathbf{n}^T M^{-1})(M\mathbf{v}) = 0 \rightarrow (M^{-T}\mathbf{n})^T (M\mathbf{v}) = 0$$
$$\rightarrow (M^{-T}\mathbf{n}) \cdot (M\mathbf{v}) = 0$$

Normalization

Unit normals may not stay unit after transformation.

- Transformation includes scale or shear
- We provide non unit normals

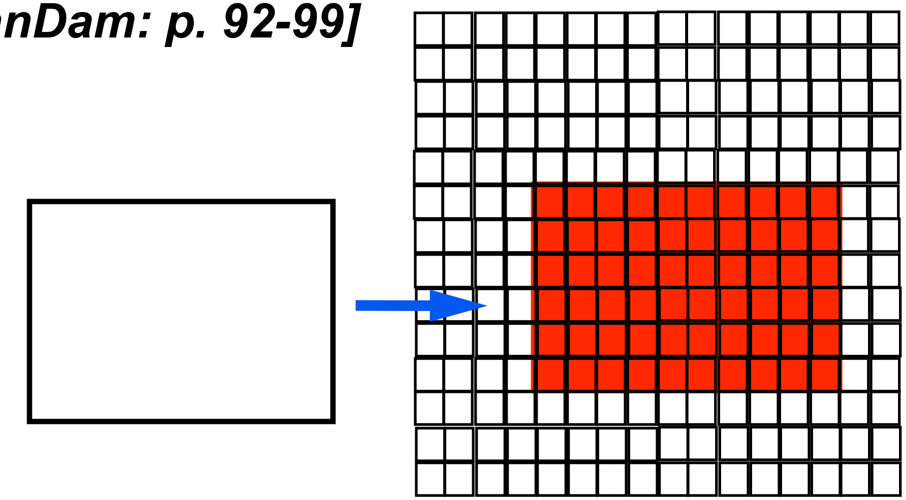
glEnable(GL_NORMALIZE) ;

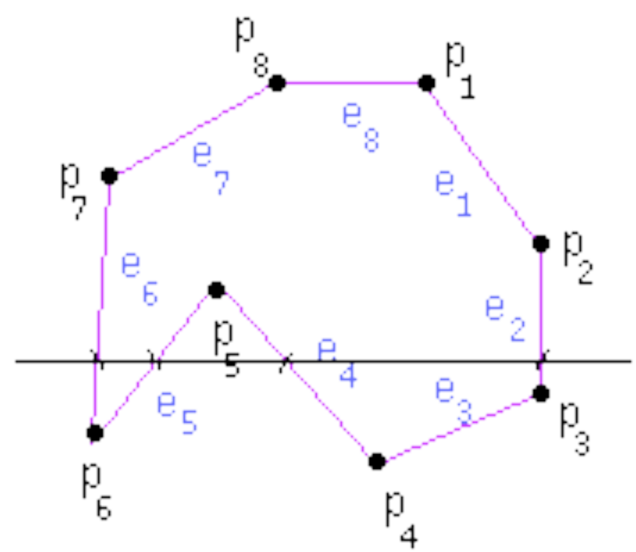
Polygons in OpenGL

```
glPolygonMode(GL_FRONT,GL_FILL) ;
glPolygonMode(GL_BACK,GL_LINE) ;
glBegin(GL_POLYGON)
glNormal3f(v1x,v1y,v1z) ;
glColor3f(r1,g1,b1) ;
glVertex3f(x1,y1,z1) ;
…
glNormal3f(vnx,vny,vnz) ;
glVertex3f(xn,yn,zn) ;
glEnd()  ;
```

Polygon Rasterization

[Hill: 570-576. Foley & vanDam: p. 92-99]





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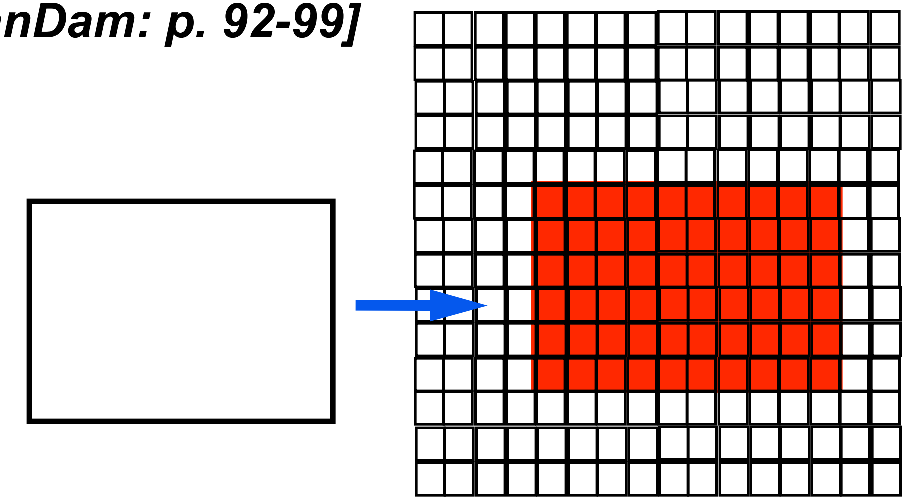
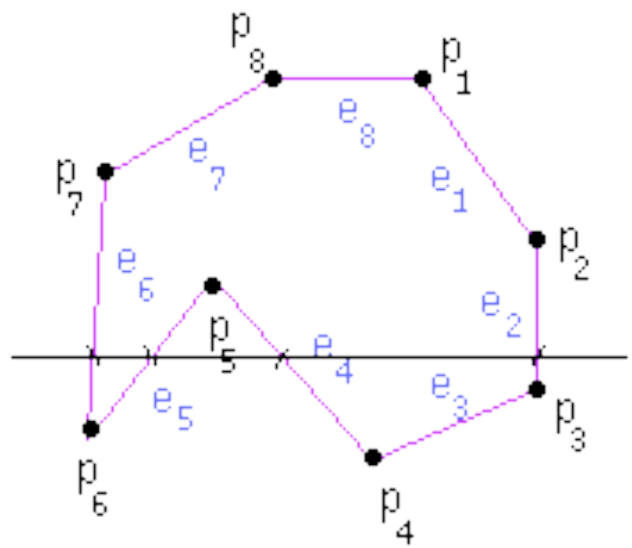


Scan conversion

shade pixels lying within a closed polygon efficiently.

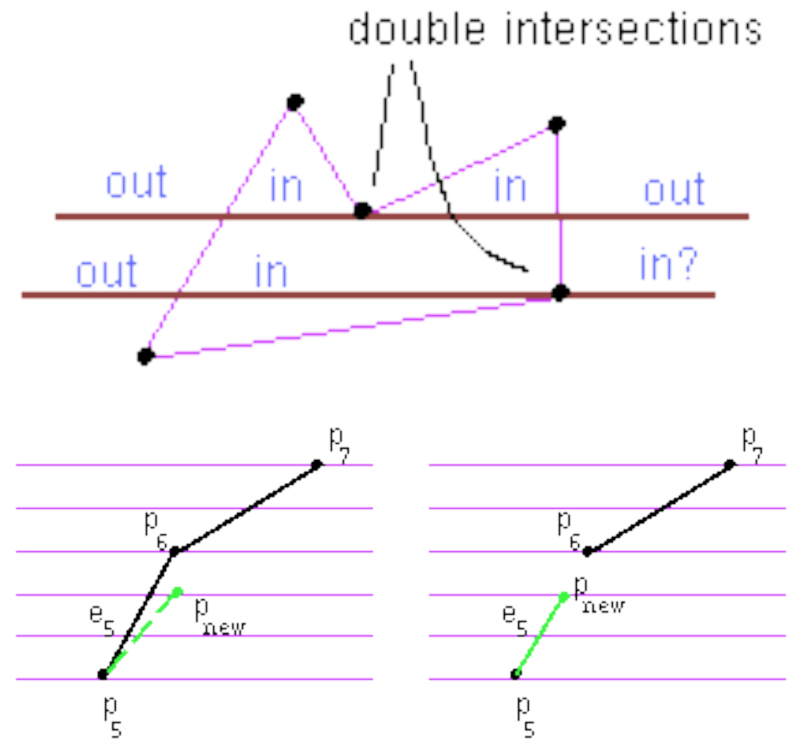
Algorithm

- For each row of pixels define a *scanline* through their centers
- intersect each scanline with all edges
- sort intersections in x
- calculate parity of intersections to determine in/out
- fill the 'in' pixels



Special cases

- Horizontal edges can be excluded
- Vertices lying on scanlines
  - *Change in sign of y_i-y_{i+1}: count twice*
  - *No change: shorten edge by one scanline*



Efficiency?

Many intersection tests can be eliminated by taking advantage of coherence between adjacent scanlines.

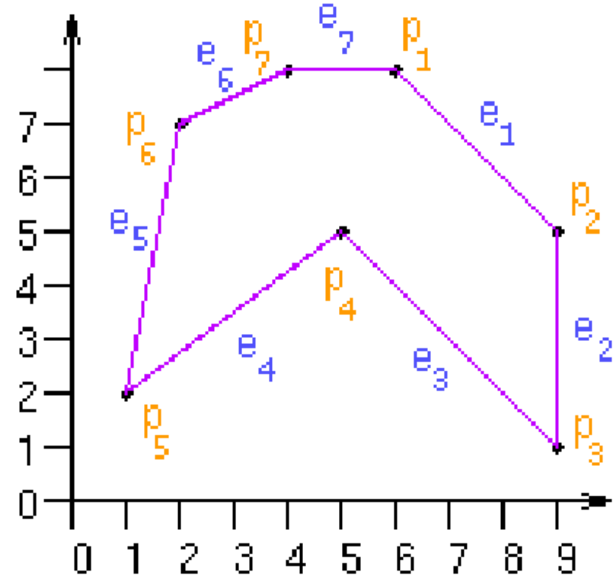
- Edges that intersect scanline y are likely to intersect y +1
- x changes predictably from scanline y to y+1

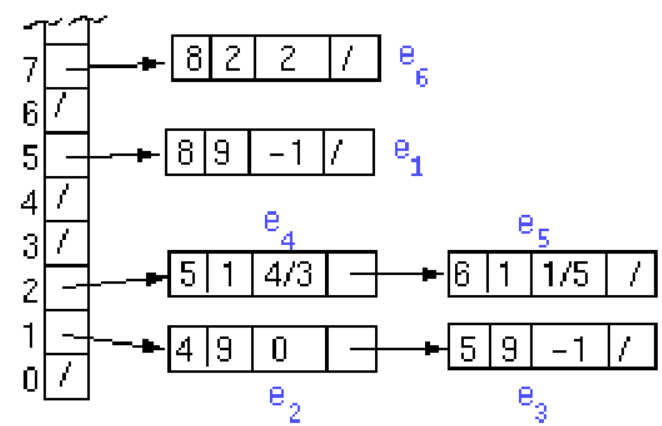
  $y = mx+a \rightarrow x = 1/m(y+a) \rightarrow x(y+1) = x(y) + 1/m$

Data structure 1: Edge table

Building edge table

- Traverse edges
- Eliminate horizontal edges
- If not local extremum, shorten upper vertex
- Add edge to linked-list for the scanline corresponding to the lower vertex, storing:
  - *y_upper: last scanline to consider*
  - *x_lower: starting x coordinate for edge*
  - *1/m: for incrementing x; compute before shortening*





Data structure 2: Active Edge List (AEL)

- The AEL is a linked list of active edges on the current scanline, y.
- Each active edge has the following information:
  - *y_upper: last scanline to consider*
  - *x: edge's intersection with current y*
  - *1/m: for incrementing x*

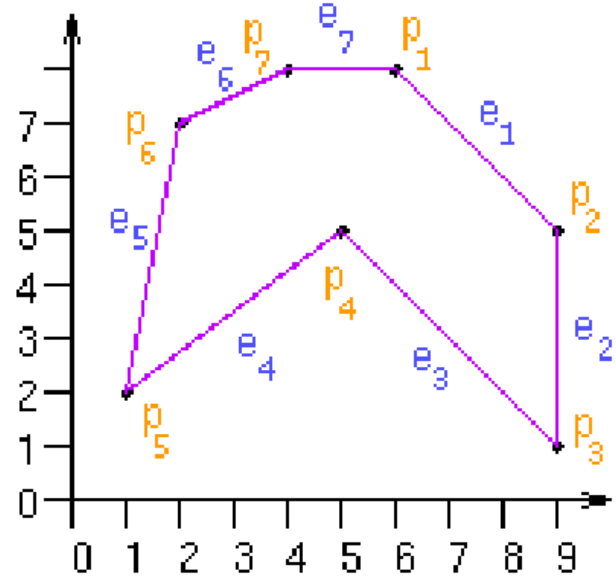
The active edges are kept sorted by x.

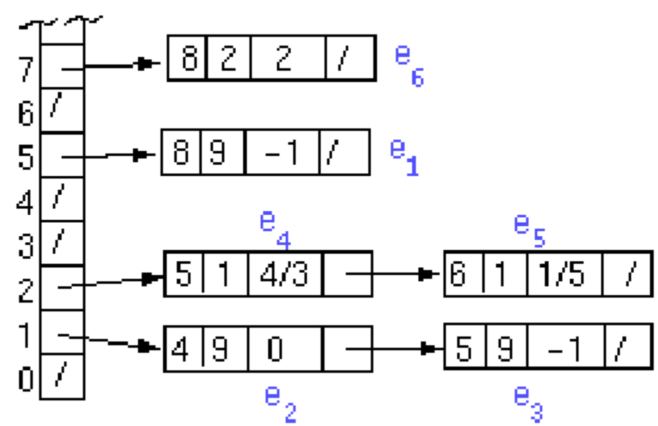
Scan conversion algorithm

```
for each scanline
    add edges in edge table to AEL
    if AEL <> NIL
            sort AEL by x
            fill pixels between edge pairs
            delete finished edges
            update each edge's x in AEL
```

Example

for each scanline
- add edges in edge table to AEL
- if AEL <> NIL
  - sort AEL by x
  - fill pixels between edge pai
  - delete finished edges
  - update each edge's x in AEL





Reminder:

Edge table

y_upper	x_lower	1/m

AEL:

y_upper	x_current	1/m

Special cases

Triangles – Convex Polygons

- Maximum two edges per scanline

Overlaping polygons

- priorities

Color, patterns

Z for visibility

Color of Interior pixels?

When no lighting is on.

- Interpolation.

When ligthing is on.

- We will see later....

Interpolating information (incrementally)

Color, Normal, Texture coordinates

Right edge (1,2):

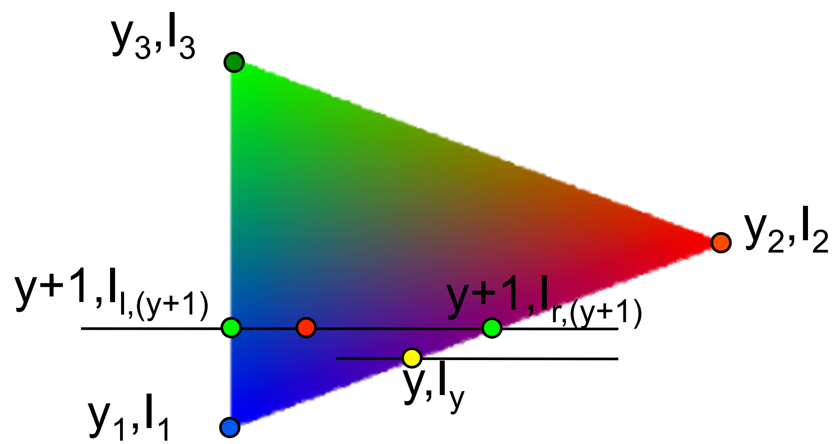
$$\frac{I_{r,(y+1)} - I_{r,y}}{(y+1) - y} = \frac{I_1 - I_2}{y_1 - y_2} \Rightarrow I_{r,(y+1)} = I_{r,y} + \frac{I_1 - I_2}{y_1 - y_2}$$

Left Edge (1,3):

$$\frac{I_{l,(y+1)} - I_{l,y}}{(y+1) - y} = \frac{I_1 - I_3}{y_1 - y_3} \Rightarrow I_{l,(y+1)} = I_{l,y} + \frac{I_1 - I_3}{y_1 - y_3}$$

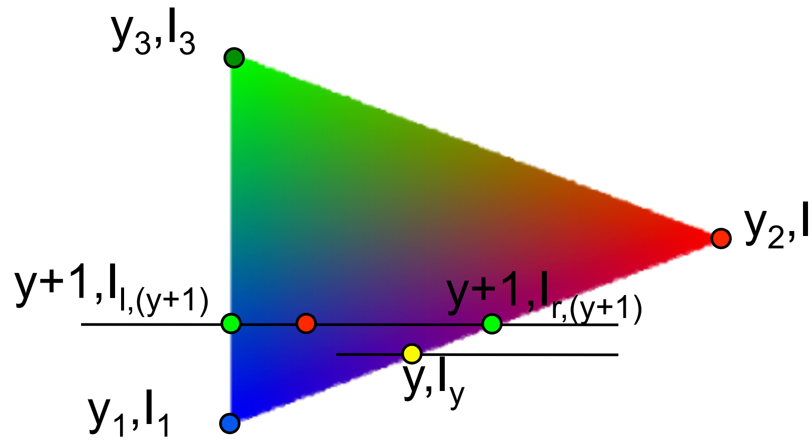
Along scanline:

$$\frac{I_{(x+1)} - I_x}{(x+1) - x} = \frac{I_r - I_l}{x_r - x_l} \Rightarrow I_{r,(y+1)} = I_{r,y} + \frac{I_r - I_l}{x_r - x_l}$$



Interpolating information (incrementally)

Color, Normal, Texture coordinates



Right edge (1,2):

$$\frac{I_{r,(y+1)} - I_{r,y}}{(y+1)-y} = \frac{I_1 - I_2}{y_1 - y_2} \Rightarrow I_{r,(y+1)} = I_{r,y} + \frac{I_1 - I_2}{y_1 - y_2}$$

Left Edge (1,3):

$$\frac{I_{l,(y+1)} - I_{l,y}}{(y+1)-y} = \frac{I_1 - I_3}{y_1 - y_3} \Rightarrow I_{l,(y+1)} = I_{l,y} + \frac{I_1 - I_3}{y_1 - y_3}$$

Along scanline:

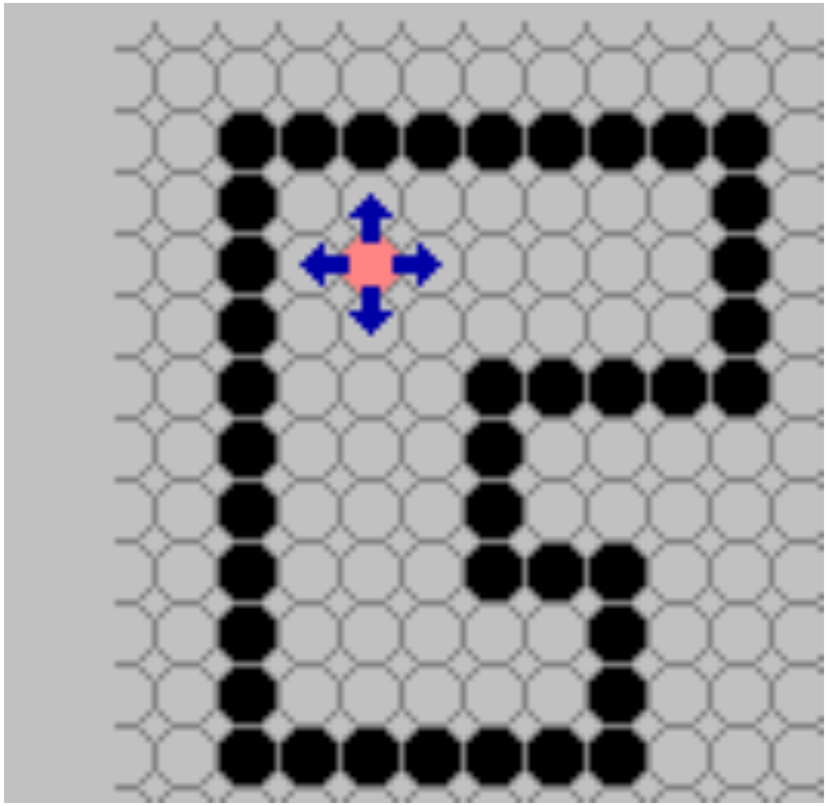
$$\frac{I_{(x+1)} - I_x}{(x+1)-x} = \frac{I_r - I_l}{x_r - x_l} \Rightarrow I_{r,(y+1)} = I_{r,y} + \frac{I_r - I_l}{x_r - x_l}$$

Constant along the line

Pixel Region filling algorithms[Hill 561-577]

Scan convert boundary

Fill in regions



2D paint programs

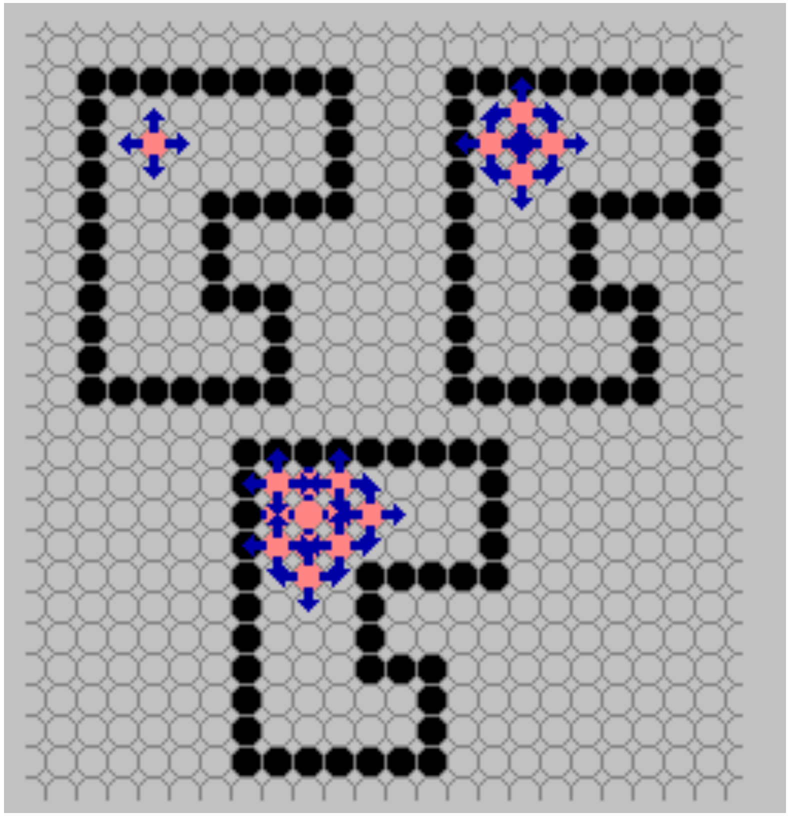
http://www.cs.unc.edu/~mcmillan/comp136/Lecture8/areaFills.html

BoundaryFill

```
boundaryFill(int x, int y, int fill, int boundary) {
        if ((x < 0) || (x >= raster.width)) return;
        if ((y < 0) || (y >= raster.height)) return;
        int current = raster.getPixel(x, y);
        if ((current != boundary) & (current != fill)) {
            raster.setPixel(fill, x, y);
            boundaryFill(x+1, y, fill,boundary);
            boundaryFill(x, y+1, fill, boundary);
            boundaryFill(x-1, y, fill, boundary);
            boundaryFill(x, y-1, fill, boundary);
    }
}
```

Flood Fill

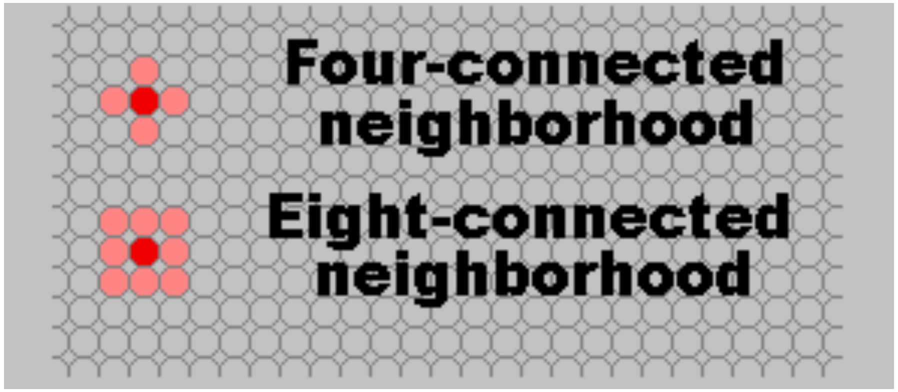
```
public void floodFill(int x, int y, int fill, int old)
{
    if ((x < 0) || (x >= raster.width)) return;
    if ((y < 0) || (y >= raster.height))  return;
    if (raster.getPixel(x, y) == old) {
                raster.setPixel(fill, x, y);
                floodFill(x+1, y, fill, old);
                floodFill(x, y+1, fill, old);
                floodFill(x-1, y, fill, old);
                floodFill(x, y-1, fill, old);
    }
}
```



Adjacency

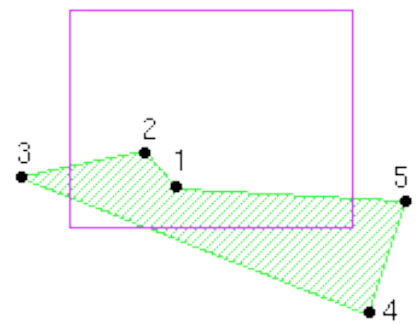
4-connected

8 connected



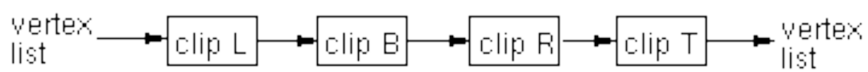
Polygon clipping (2D)[Hill 181-208]

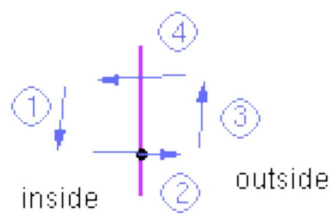
Sutherland-Hodgeman [Hill 202]



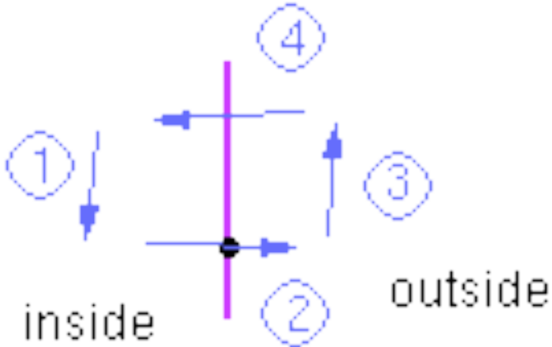
for each side of clipping window

- for each edge of polygon
    - output points based upon the following table

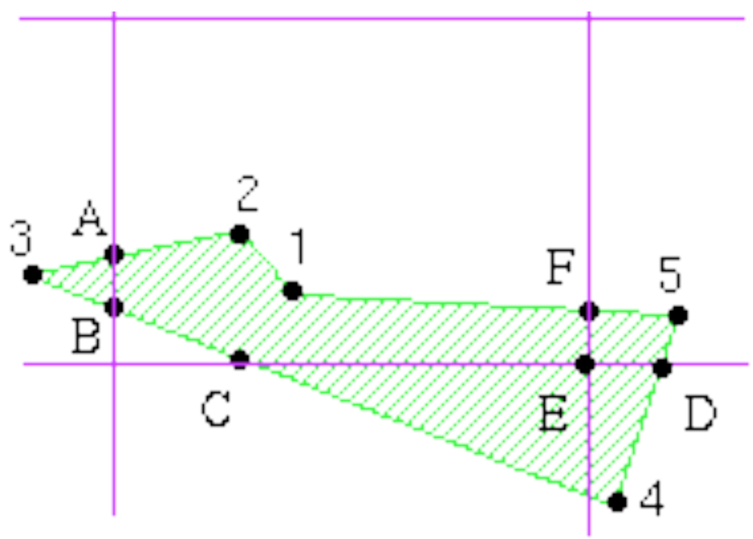




case #	first point	second point	output point(s)
1	inside	inside	second point
2	inside	outside	intersection point
3	outside	outside	none
4	outside	inside	intersection point and second point



case #	first point	second point	output point(s)
1	inside	inside	second point
2	inside	outside	intersection point
3	outside	outside	none
4	outside	inside	intersection point and second point



original: 1,2,3,4,5,1

clip L: 1,2,A,B,4,5,1
clip B: 1,2,A,B,C,D,5,1
clip R: 1,2,A,B,C,E,F,1
clip T: (same)

Outcodes for trivial reject/accept

[Hill 389] A vertex outcode consists of four bits: TBRL, where:

T is set if y > top,

B is set if y < bottom,

R is set if x > right, and

L is set if x < left.

Trivial accept: all vertices are inside (all outcodes are 0000, bitwise OR)

Trivial reject: all vertices are outside with respect to any given side (bitwise AND is not 0000)