Lecture Notes (Shinnerl)

Nonsquare LU Factorizations and Rank Determination

LU factorization and rank determination

Let $A \in \mathbf{R}^{m\times n}$ with $r \equiv \text{rank}(A)$.

The LU factorization *with row interchanges* always exists: $PA = LU$; however, at some stage, the entire leading column of the remaining matrix may be zero (approximately).

Example: Suppose, after the first stage of elimination,

$$A^{(2)} = \begin{pmatrix} 7 & 6 & 7 & 3 & 3 \\ 0 & \boxed{0} & 5 & 3 & 5 \\ 0 & 0 & 6 & 2 & 0 \\ 0 & 0 & 5 & 2 & 3 \end{pmatrix}.$$

How should we proceed, if we wish to determine rank(A)?

$$A^{(2)} = \begin{pmatrix} 7 & 6 & 7 & 3 & 3 \\ 0 & \boxed{0} & 5 & 3 & 5 \\ 0 & 0 & 6 & 2 & 0 \\ 0 & 0 & 5 & 2 & 3 \end{pmatrix}.$$

If we simply skip elimination in the second column and proceed to the third column, we obtain

$$A^{(4)} = U = \begin{pmatrix} 7 & 6 & 7 & 3 & 3 \\ 0 & \boxed{0} & 5 & 3 & 5 \\ 0 & 0 & 6 & 2 & 0 \\ 0 & 0 & 0 & 1/3 & 3 \end{pmatrix}, \quad \text{but} \quad \text{rank}(U) = 4.$$

The fact that $u_{22} = 0$ does **not** indicate rank deficiency in this case!

Remark. The rank of an upper-triangular matrix U is *not always* simply the number of nonzero diagonal elements in U.
Example:

$$U_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The number of nonzero diagonals in U_2 is 0, but $\text{rank}(U_2) = 1$.

Similarly, consider

$$U_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}.$$

The number of nonzero diagonals is 0, but $\text{rank}(U_n) = n - 1$.

Rank-determination strategy: Rather than skip elimination when a zero column is encountered, use a column permutation to enable the elimination to proceed:

$$A^{(2)} = \begin{pmatrix} 7 & 6 & 7 & 3 & 3 \\ 0 & \boxed{0} & 5 & 3 & 5 \\ 0 & 0 & 6 & 2 & 0 \\ 0 & 0 & 5 & 2 & 3 \end{pmatrix}; \quad A^{(2)} P_c^{(2)^T} = \begin{pmatrix} 7 & 3 & 7 & 3 & 6 \\ 0 & \boxed{5} & 5 & 3 & 0 \\ 0 & 0 & 6 & 2 & 0 \\ 0 & 3 & 5 & 2 & 0 \end{pmatrix};$$

$$A^{(3)} = \begin{pmatrix} 7 & 3 & 7 & 3 & 6 \\ 0 & 5 & 5 & 3 & 0 \\ 0 & 0 & 6 & 2 & 0 \\ 0 & 0 & 2 & 0.2 & 0 \end{pmatrix}; \quad U = A^{(4)} = \begin{pmatrix} 7 & 3 & 7 & 3 & 6 \\ 0 & 5 & 5 & 3 & 0 \\ 0 & 0 & 6 & 2 & 0 \\ 0 & 0 & 0 & -7/15 & 0 \end{pmatrix}.$$

Now $U \equiv A^{(4)}$ contains a square full-rank upper-triangular matrix; hence, $\text{rank}(U) = 4$ is clear.

Rank-Revealing Form of the LU Factorization

If *column* permutations are *also* used, then a nonzero pivot can be found at every stage until the entire remaining matrix is zero. The LU factorization takes the following form.

$$P_r A P_c^T = LU = \begin{pmatrix} L_{11} & 0 \\ L_{21} & I_{m-r} \end{pmatrix} \begin{pmatrix} U_1 \\ 0 \end{pmatrix},$$

where L_{11} is $r \times r$ unit lower triangular, and U_1 is $r \times n$ upper triangular.

It follows immediately that rank$(A) =$ rank(U_1).

```
A =
    48    16    46    32    26
     6     2     6     5     3
    30    10    39    61     6
    36    12    35    26    19
```

Example: find rank(A).

```
Pr*A*Pc' =
    48    32    46    16    26
    30    61    39    10     6
     6     5     6     2     3
    36    26    35    12    19
```

```
L =
     1     0     0     0
   5/8     1     0     0
   1/8  1/41     1     0
   3/4  2/41     0     1
```

```
U =
    48    32    46    16    26
     0    41  41/4     0 -41/4
     0     0     0     0     0
     0     0     0     0     0
```

Hence, rank$(A) = 2$.

Permutations of Linear Systems of Equations $Ax = b$

Permuting the rows of A requires a corresponding permutation in the right-hand side.

$$Ax = b \quad \Longleftrightarrow \quad P_r A x = P_r b.$$

Permuting the columns of A requires a corresponding permutation in the variables x.

$$Ax = b \quad \Longleftrightarrow \quad (A P_c^T)(P_c x) = b.$$

Hence, given the factorization $P_r A P_c^T = LU$, to solve

$$Ax = b \quad \Longleftrightarrow \quad (P_r A P_c^T)(P_c x) = P_r b \quad \Longleftrightarrow \quad LU(P_c x) = P_r b,$$

we must solve the following systems in the following order:

$$Ly = P_r b, \quad Uz = y, \quad P_c x = z.$$

Solution of Underdetermined Systems by LU Factorization

$P_r A P_c^T = LU$ for $A \in \mathbf{R}^{m\times n}$ with $m < n$ ("short and fat" A)

$$P_r A P_c^T = L\,(U_1\ U_2)\,,$$

where $L \in \mathbf{R}^{n\times n}$, $U_1 \in \mathbf{R}^{m\times m}$, and $U_2 \in \mathbf{R}^{m,n-m}$.
U has the same shape as A: $U \in \mathbf{R}^{m\times n}$.

Example ($P_r = I_3$, $P_c^T = I_5$).

```
A =                                L =                  U =

    6    5    0    3    0             1    0    0           6    5    0    3    0
   12   15    4    8    6             2    1    0           0    5    4    2    6
   18   30   10   13   16             3    3    1           0    0   -2   -2   -2
```

continued $\rightarrow$

Solving $Ax = b$ for $A \in \mathbf{R}^{m\times n}$ with $m < n$ ("short and fat" A)

$$P_r A P_c^T = L\,(U_1\ U_2)\,,$$

The partition of U induces a corresponding partition of $z = P_c x$,

$$z = P_c x = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Suppose A has full row rank (rank(A) $= m$), so that U_1 is non-singular. Then, if $b \in$ range(A), each arbitrary choice of z_2 determines a unique z_1 such that $Ax = AP_c^T z = b$. The components of z_2 can be used as parameters.

continued $\rightarrow$

Solving $Ax = b$ by $P_r A P_c^T = LU$ for $A \in \mathbf{R}^{m\times n}$ with $m < n$

$$Ax = b \Longleftrightarrow P_r A P_c^T P_c x = L\,(U_1 \; U_2)\, P_c x = P_r b.$$

Let $z = P_c x$. Then

$$P_r A P_c^T P_c x = L\,(U_1 \; U_2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = L\,(U_1 z_1 + U_2 z_2) = P_r b,$$

and hence

$$L U_1 z_1 = P_r b - L U_2 z_2. \tag{1}$$

To solve $Ax = b$, we can set z_2 arbitrarily, solve (1) for z_1, and then set $x = P_c^T z$. E.g., for the data on page 9, if $b = (1, 1, -2)^T$, let $x_2 = (0, 0)^T$, then $x = (1, -1, 1, 0, 0)^T$.

Exercise:
Derive an analogous process for $Ax = b$ when rank$(A) < m < n$.

$PA = LU$ **for** $A \in \mathbf{R}^{m\times n}$ **with** $m > n$ **("tall and thin"** A**).**
Assume that rank(A) $= n$. Then

$$PA = L\begin{pmatrix} U_1 \\ 0 \end{pmatrix} = (L_1\ L_2)\begin{pmatrix} U_1 \\ 0 \end{pmatrix} = L_1 U_1,$$

where $L \in \mathbf{R}^{m\times m}$, $L_1 \in \mathbf{R}^{m\times n}$, $U_1 \in \mathbf{R}^{n\times n}$, and $0 \in \mathbf{R}^{m-n,n}$.
U has the same shape as A: $U \in \mathbf{R}^{m\times n}$.

Example ($P = I_5$).

```
A =                 L =                           U =
    7    3    4        1    0    0    0    0         7    3    4
    7    9    4        1    1    0    0    0         0    6    0
   56   36   34        8    2    1    0    0         0    0    2
   63    9   42        9   -3    3    1    0         0    0    0
   42   60   30        6    7    3    0    1         0    0    0
```

Least-Squares Approximation by LU Factorization

Assume that $\text{rank}(A) = n$. The factorization

$$PA = L \begin{pmatrix} U_1 \\ 0 \end{pmatrix}$$

does not help us directly to *minimize* $\|Ax - b\|$, but we can use it with the following result to obtain a useful x.

Exercise. $\|Ax - b\| \leq \|L\|\|Ux - y\|$, where $Ly = Pb$, and $\|L\| \leq m/\sqrt{2}$.

Hence, minimizing $\|Ux - y\|$ typically gives a useful x for making $\|Ax - b\|$ small. Write $y = (y_1, y_2)^T$. Since

$$\|Ux - y\|^2 = \|U_1 x - y_1\|^2 + \|y_2\|^2,$$

solving $U_1 x = b_1$ minimizes $\|Ux - y\|^2$.

Numerical Stability of $PA = LU$

Main Result: The factorization is stable, as long as $\|U\|$ is not too large compared to $\|A\|$.

Definition For $x \in \mathbf{R}^n$, $\|x\|_\infty = \max_i |x_i|$.
Exercise: For $A \in \mathbf{R}^{m\times n}$, $\|A\|_\infty \equiv \max_i \sum_j \left|a_{ij}\right|$.

Definition ("growth factor") Let $g_{pp} = \frac{\max |u_{ij}|}{\max |a_{ij}|}$.

continued $\rightarrow$

Numerical Stability of $PA = LU$

Theorem Suppose $A \in \mathbf{R}^{n \times n}$, and let $\widehat{L}$ and $\widehat{U}$ denote the computed LU factors of A by Gaussian Elimination with partial pivoting (row interchanges only). Then

$$\widehat{L}\widehat{U} = P(A + \Delta A),$$

with

$$\frac{\|\Delta A\|_\infty}{\|A\|_\infty} \leq 3 g_{pp} n^3 \epsilon,$$

where ϵ denotes machine precision ($\approx 10^{-16}$)

Instability in $PA = LU$ is Rare but Possible

In practice, the growth factor g_{pp} is almost always less than n. However, the following example shows that it *can* be as large as 2^{n-1} for specially structured matrices.

$$W_n = \begin{pmatrix} 1 & & & & 1 \\ -1 & 1 & & & 1 \\ -1 & -1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 1 \end{pmatrix}.$$

Therefore, the growth factor must be checked to ensure stability of the factorization.