Winter 2006　　EE 103 (Shinnerl)　　Assignment 4

Partial Solutions.

Where solutions are given, please either substitute your own solution or rewrite the given solution yourself with some change of notation.

Exercise 1. Let $V = \{v_1, \ldots, v_m\}$ be some arbitrary subset of vectors taken from a vector space $\mathcal{W}$.

(a) Show explicitly that a scalar multiple of a linear combination of the vectors in V is also a linear combination of vectors in V.

Solution. *By linearity of scalar-vector multiplication, for every scalar* β*,*

$$\beta \cdot \sum_{i=1}^{m} \alpha_i v_i = \sum_{i=1}^{m} \beta \alpha_i v_i = \sum_{i=1}^{m} (\beta \cdot \alpha_i) v_i,$$

a linear combination with coefficients $\beta\alpha_i$*.*

(b) Show explicitly that a linear combination of linear combinations of vectors in V also a linear combination of vectors in V.

Solution.

$$\sum_j \beta_j \left(\sum_{i=1}^{m} \alpha_i v_i \right) = \sum_j \sum_{i=1}^{m} \beta_j \alpha_i v_i = \sum_{i=1}^{m} \alpha_i (\sum_j \beta_j) v_i,$$

which is a linear combination of the v_i*, with* i*th coefficient* $\alpha_i \sum_j \beta_j$*.*

(c) Use (a) and (b) above to show that span(V) is a vector subspace of $\mathcal{W}$.

Solution. *As always, it suffices to show that span(*$\mathcal{V}$*) is closed under vector addition (shown in Part (b) above) and scalar multiplication (shown in Part (a) above).*

(d) Now suppose $\mathcal{W} = \mathbf{R}^n$; hence, $v_i \in \mathbf{R}^n$ for each $v_i \in V$. Define a matrix A with the property that range(A) = span(V). How are the dimensions and rank of A related to V, span(V), and $\mathcal{W}$?

Exercise 2.

(a) What is the *inverse* of a matrix?

(b) What shape of matrix can have an inverse?

(c) Prove that the inverse of a matrix is unique, if it exists.

Solution. *Suppose* B *and* C *are both inverses of* A*. Then* $I = AB = AC$*. Subtracting* AC *from both sides of the last equation gives* $A(B - C) = 0$*, since matrix-matrix multiplication is linear. Because* A *has independent columns, every column of the matrix* $B - C$ *must be zero. Hence,* $B = C$*.*

(d) Define the *transpose* A^T of a matrix $A \in \mathbf{R}^{m \times n}$.

Solution. $A^T_{ij} \equiv A_{ji}$ *for all* $i \in \{1, \ldots n\},\ j \in \{1, \ldots, m\}$*.*

(e) Prove that if the matrix-matrix product AB is conformable (i.e., the number of columns of A equals the number of rows of B), then $(AB)^T = B^T A^T$. Verify this statement in Matlab for $m \times p$ matrix A and $p \times n$ matrix B, for a few choices of A and B, both square ($m = p = n$) and non-square.

Solution. *The* ij*th component of* $(AB)^T$ *equals the* ji*th component of* AB *equals the inner product of the* j*th row of* A *and the* i*th column of* B*. This is the same as the* ij*th component of* $B^T A^T$*.*

(f) Prove that if A is invertible, then so is A^T, and $(A^{-1})^T = (A^T)^{-1}$. Explain why this fact justifies the use of the notation A^{-T}. Verify this statement in Matlab for a few nontrivial choices of A.

Solution Hint. *Use the uniqueness of inverses already shown above.*

Exercise 3.

(a) Show explicitly that matrix-vector multiplication is a linear operation. For scalar α, matrix $A \in \mathbf{R}^{m\times n}$, and vectors $x, y \in \mathbf{R}^n$, $A(\alpha x) = \alpha Ax$ and $A(x+y) = Ax + Ay$.

Solution. *Use the definition of matrix multiplication: if* $y = Ax$, *then* $y_i = \sum_{j=1}^n a_{ij}x_j$ *for each* $i = 1, \ldots, m$.

$$(A(\alpha x))_i = \sum_{j=1}^n \alpha a_{ij}x_j = \alpha \sum_{j=1}^n a_{ij}x_j = \alpha(Ax)_i$$

Hence, $A(\alpha x) = \alpha Ax$.

$$(A(x+y))_i = \sum_{j=1}^n a_{ij}(x_j + y_j) = \sum_{j=1}^n a_{ij}x_j + \sum_{j=1}^n a_{ij}y_j = (Ax + Ay)_i.$$

Hence, $A(x+y) = Ax + Ay$.

(b) Show that if $F : \mathbf{R}^n \to \mathbf{R}^m$ is a linear function and $G : \mathbf{R}^m \to \mathbf{R}^p$ is a linear function, then the composition function $H : \mathbf{R}^n \to \mathbf{R}^p$ defined by $H(x) = G(F(x))$ is also a linear function. **Remark.** This result says that any sequence of linear transformations (2 or more) may be viewed as a single linear transformation.

(c) If, in the preceding question, $m = n = p$ and F and G are invertible, then show that H is invertible, and $H^{-1}(y) = F^{-1}(G^{-1}(y))$. Hint: to show that functions S and T are inverses of each other, it suffices to show that (i) $T(S(x)) = x$ for all x in the domain of S, and (ii) $S(T(y)) = y$ for all y in the domain of T.

Verify the following statement in Matlab for a few nontrivial choices of A and B.
Corollary. If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.

Exercise 4.

Prove the following and illustrate in Matlab for a few nontrivial choices of the given matrices.

(a) If D and E are diagonal, then $DE = ED$.

Because D *is diagonal,* $d_{ij} = 0$ *for* $i \neq j$. *Show* DE *and* ED *are also diagonal.*

(b) Premultiplication of $A \in \mathbf{R}^{m\times n}$ by a diagonal matrix $D \in \mathbf{R}^m$ scales the ith row of A by d_{ii}.

Use the row-oriented view of matrix multiplication from the lecture notes (or don't).

(c) Postmultiplication of $A \in \mathbf{R}^{m\times n}$ by a diagonal matrix $D \in \mathbf{R}^n$ scales the jth column of A by d_{jj}.

Use the column-oriented view of matrix multiplication from the lecture notes (or don't).

Exercise 5.

Let $V = \{v_1, v_2, \ldots, v_m\}$ be a finite set of vectors taken from a vector space $\mathcal{W}$. Let $\bar{V}$ be the same as V, except that, in $\bar{V}$, some element $v_i \in V$ is replaced by $v_i + \kappa_j v_j$, with $i \neq j$ and $\kappa_j \neq 0$. I.e.,

$$\bar{V} = \{v_1, \ldots, v_{i-1}, v_i + \kappa_j v_j, v_{i+1}, \ldots, v_m\}.$$

Show that $\text{span}(V) = \text{span}(\bar{V})$.

Exercise 6. Basic Theory.

Hints for this exercise are posted separately.

Now let $\{v_1, v_2, \ldots, v_m\}$ be the rows of the matrix A. (Note that A has m rows.) Gaussian elimination with row interchanges (GEPP — "pp" stands for "partial pivoting") can be used to reduce A to a matrix E in *row-echelon form*, defined as follows.

(i) All zero-rows of E are at the bottom. I.e., if the ith row of E is zero for $i < m$, then each row j below it, i.e., with $j > i$, is also zero.

(ii) If e_{ij} is the first nonzero element in the ith row of E, then all elements below it and to its left (i.e. all e_{pq} such that $p > i$ and $q \leq j$) are zero.

The first nonzero entry in each nonzero row of E is called a *pivot.* A column or row containing a pivot is called a *pivot column* or row, respectively. Here is an example of a matrix in row echelon form. The symbol × stands for any value, zero or nonzero. The symbol $*$ stands for a nonzero pivot value.

$$E = \begin{pmatrix} * & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ 0 & 0 & * & \times & \times & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & * & \times & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & \times & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

To obtain *reduced row-echelon form* from row-echelon form, (i) use row operations to eliminate all elements in each pivot column except the pivot itself, then (ii) scale the rows (a linear operation — use a diagonal matrix) to make the pivots all equal one. After reducing E above to reduced row-echelon form, we obtain the following matrix $\bar{E}$.

$$\bar{E} = \begin{pmatrix} 1 & \times & 0 & 0 & \times & \times & \times & 0 & \times & 0 \\ 0 & 0 & 1 & 0 & \times & \times & \times & 0 & \times & 0 \\ 0 & 0 & 0 & 1 & \times & \times & \times & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \times & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

(a) (i) Prove that the nonzero rows of a matrix in reduced row-echelon form are linearly independent.
(ii) Prove that the pivot columns of a matrix in reduced row-echelon form are linearly independent.

(b) Prove that interchanging 2 rows of any $m \times n$ matrix A is an invertible linear operation. Give the form of the matrix P that exchanges rows i and j of A in the product PA (hint: interchange rows i and j of the identity matrix).

(c) Each step of the GEPP reduction is either an interchange of two rows or the replacement of some row v_i by the vector $v_i + \kappa_j v_j$ for some other row v_j and some nonzero scalar κ_j, as in the previous exercise. Show that each such row replacement is an invertible linear operation.

(d) By a previous exercise, the entire sequence of row replacements and row interchanges can be viewed as a single, invertible, linear transformation, R, applied to A on its left. Explain why the row space of $E = RA$ is the same as the row space of A.

(e) Prove that the nonzero rows of the (either reduced or unreduced) row-echelon form of any matrix A are a *basis* for the row space of A.

(f) Prove that the pivot positions in the row-echelon form of A are uniquely determined. I.e., no matter what row ordering of A you use, the pivots in its row-echelon form must always take the same positions (row and column indices). (Hint: show that changing pivot positions changes the row space.)

(g) Explain how the above results show that

(i) The (maximum possible) number of linearly independent rows of a matrix (i.e., *row rank*) equals the (maximum possible) number of linearly independent columns (i.e., the *column rank*) of the matrix. Both these numbers are equal to the number of vectors in a basis for the range of the matrix (i.e., the *rank* of the matrix).

(ii) Any 2 bases for the same subspace S have the same number of elements (hint: let S be the row space of a matrix).

(iii) The (unmodified) rows in A corresponding to nonzero rows in the row-echelon form of A are also a basis for the row space of A.

(h) Show by example that, in general, $\mathrm{range}(E) \neq \mathrm{range}(A)$.

(i) Show that the columns of A corresponding to pivot columns of E define a basis for $\mathrm{range}(A)$.

Exercise 7.

Given fixed $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, what is the significance each of the following facts in relation to the linear system of equations $Ax = b$?

(a) A has full row rank; i.e., $\mathrm{rank}(A) = m$.

(b) A has full column rank; i.e., $\mathrm{rank}(A) = n$.

Solution Hints.

(a) *System $Ax = b$ has a solution if and only if $b \in \mathrm{range}(A)$. It was shown in lecture that* $\mathrm{range}(A)$ *is a subspace of $\mathbf{R}^m$. By definition,* $\mathrm{rank}(A) = \dim(\mathrm{range}(A)$. *If $b \in \mathbf{R}^m$ and $b \notin \mathrm{range}(A)$, then* $\dim(\mathbf{R}^m) > \dim(\mathrm{range}(A))$.

(b) *If* $\mathrm{rank}(A) = n$, *then the columns of A are linearly independent.*

Exercise 8.

Explain the more efficient way to calculate the given matrix-vector products.

(a) $y = (AB)x$ or $y = A(Bx)$.

(b) $A = (I + uv^T)$ followed by $y = Ax$ or $y = x + (v^T x)u$.

Solution Hint. *Just count the total number of multiplications and additions in each case.*

Exercise 9.

Let

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$

where $A_1 \in \mathbf{R}^{m_{11} \times n_{11}}$, $B_1 \in \mathbf{R}^{m_{11} \times n_{12}}$, $C_1 \in \mathbf{R}^{m_{12} \times n_{11}}$, $D_1 \in \mathbf{R}^{m_{12} \times n_{12}}$; and $A_2 \in \mathbf{R}^{m_{21} \times n_{21}}$, $B_2 \in \mathbf{R}^{m_{21} \times n_{22}}$, $C_2 \in \mathbf{R}^{m_{22} \times n_{21}}$, $D_2 \in \mathbf{R}^{m_{22} \times n_{22}}$. State the conditions on $m_{11}, \ldots, n_{22}$ under which the expression

$$M = \begin{pmatrix} A_1 A_2 + B_1 C_2 & A_1 B_2 + B_1 D_2 \\ C_1 A_2 + D_1 C_2 & C_1 B_2 + D_1 D_2 \end{pmatrix}$$

is well defined. Show explicitly that, under these conditions, M is a valid representation of $M_1 M_2$. Verify in Matlab for 3 separate particular examples.

Solution. *Well defined if and only if*

$$n_{11} = m_{21} \quad \textit{and} \quad n_{12} = m_{22}.$$

Consider M_{ij} for, e.g., $m_{11} < i < m_{12}$ and $j < n_{11}$. (This is the $(2,1)$ block of M). Recall that M_{ij} is the dot product of the ith row of M_1 with the jth column of M_2. Hence, if $(\tilde{c}^{(1)}_{i-m_{11}})^T$ denotes the $i - m_{11}$th row of C_1 and similarly for $(\tilde{d}^{(1)}_{i-m_{11}})^T$, then

$$M_{ij} = ((\tilde{c}^{(1)}_{i-m_{11}})^T, (\tilde{d}^{(1)}_{i-m_{11}})^T) \begin{pmatrix} a^{(2)}_j \\ c^{(2)}_j \end{pmatrix},$$

where $a^{(2)}_j$ is the jth column of A_2, and similarly for $c^{(2)}_j$. This expression matches the corresponding entry in $C_1 A_2 + D_1 C_2$. The other cases for i and j are similar.

Exercise 10.

(This exercise is a special case of the preceding one.) Show that a matrix-matrix product can always be viewed as a sum of outer products. Suppose $A \in \mathbf{R}^{m \times k}$ and $B \in \mathbf{R}^{k \times n}$. Write

$$A = \begin{pmatrix} | & & | \\ a_1 & \ldots & a_k \\ | & & | \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} — & \tilde{b}_1^T & — \\ & \vdots & \\ — & \tilde{b}_k^T & — \end{pmatrix}.$$

That is, a_q is the qth column of A, and $\tilde{b}_q^T$ is the qth row of B. Show that

$$AB = \sum_{q=1}^{k} a_q \tilde{b}_q^T.$$

Verify in Matlab for a few particular examples. *Hint:* It suffices to consider the ijth entry of both sides.
Solution. *By definition,*

$$(AB)_{ij} = \sum_{p=1}^{k} a_{ip} b_{pj}.$$

For each $p \in \{1, \ldots, k\}$, *interpret* a_{ip} *as the ith element of the pth column of A, and interpret* b_{pj} *as the jth element of the pth row of B:*

$$(AB)_{ij} = \sum_{p=1}^{k} a_{ip} b_{pj} = \sum_{p=1}^{k} (a_p)_i (\tilde{b}_p^T)_j = \sum_{p=1}^{k} (a_p \tilde{b}_p^T)_{ij} = \left(\sum_{q=1}^{k} a_q \tilde{b}_q^T \right)_{ij}.$$

Alternatively, since each product $a_q \tilde{b}_q^T$ *is conformable (well defined dimensionally), the result follows by simple block matrix multiplication, where the blocks* A_q *of A are the columns of A, and the blocks* B_q *of B are the rows of B:*

$$\begin{pmatrix} A_1 & A_2 & \ldots & A_k \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{pmatrix} = A_1 B_1 + A_2 B_2 + \cdots + A_k B_k.$$

In this case, each $A_q B_q$ *happens to be an "outer product"* $a_k \tilde{b}_k^T$, *but the result is true for any partitions of A and B such that each product* $A_q B_q$ *is conformable.*

Exercise 11. Given $A \in \mathbf{R}^{m \times n}$ with $m < n$, $b \in \mathbf{R}^m$, and $x_0 \in \mathbf{R}^n$ such that $Ax_0 = b$. Let the columns of Z form a basis for the null space of A. Show that

$$\{x \,|\, Ax = b\} \equiv \{x_0 + Zw \,|\, w \in \mathbf{R}^{n-r}\},$$

where $r = \text{rank}(A)$. *Hint.* To show two sets are equal, you must show that each is a subset of the other.
Partial Solution. *If* $Ax = Ay = b$, *then* $A(x - y) = 0$. *I.e.,* $x - y \in \text{null}(A)$, *and hence by definition of Z, there exists some vector v such that* $y - x = Zv$. *Let* $x = x_0$ *and hold it fixed while y takes on different solutions. Hence,*

$$\{x \,|\, Ax = b\} \subset \{x_0 + Zw \,|\, w \in \mathbf{R}^{n-r}\}.$$

You must still show that the right-hand expression is a subset of the left-hand one.

Exercise 12. Specify whether each of the following operations on a vector $x \in \mathbf{R}^n$ is linear or nonlinear. Prove your answers.

To show that an operation $f(x)$ *is linear in x, you have to prove that* $f(\alpha x) = \alpha f(x)$ *and* $f(x + y) = f(x) + f(y)$. *But to show non-linearity, you have to only find one counterexample to one of these two properties.*

(a) Find a weighted average of the elements of x (the weights are constants).

A weighted average has the form $f(x) = \sum_i \alpha_i x_i$ *with* $\sum_i \alpha_i = 1$. $\boxed{\textit{LINEAR}}$

(b) Find the maximum element of x.

Consider $x = (2, 1)$ *and* $y = (-2, 0)$. $\boxed{\textit{NONLINEAR}}$

(c) Reverse the order of the elements of x.

See the lecture notes for the 2×2 *case and extend it to* $\mathbf{R}^n$. ***Hint.*** *Consider the "antidiagonal" matrix*

$$m_{ij} = \begin{cases} 1 & \textit{if } i + j = n + 1 \\ 0 & \textit{otherwise} \end{cases}$$

I.e.,

$$M = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

(d) Replace each element of x by its absolute value.

Exercise 13. Let

```
A =                                                                  b =
    0.8462    0.0196    0.7095    0.6822    0.3784                       1.0000
    0.5252    0.6813    0.4289    0.3028    0.8600                       2.0000
    0.2026    0.3795    0.3046    0.5417    0.8537,                      3.0000
```

Use MATLAB (or Octave) to give concise descriptions of the following sets.
Solution Hint. *Matlab has commands* `orth` *and* `null` *for determining bases for the range and nullspace. You should also check that columns of A corresponding to pivot columns in* `rref(A)` *form a basis for* range(A)*. For part (c), use the null-space characterization of the solution set to a linear system derived in the related Exercise above.* **(a)** range(A). **(b)** null(A). **(c)** $\{x \mid Ax = b\}$. **(d)** range(A^T). **(e)** null(A^T).