May 1, 2019

Instructions

- Your solutions are due at 11:59 pm on May 10, 2019. Please typeset your work and submit the resulting PDF document on CCLE using the TurnItIn feature. Late work will not be accepted or graded, resulting in zero credit for this assignment.

- You cannot collaborate with other students on this assignment. The work that you submit must be your own.

- This exam can be solved in its entirety using the course material taught so far, without consulting any additional sources. However, you are welcome to use any scholarly sources, including your textbook and the Internet. If you do, please write the solution in your own words and acknowledge the sources that you have consulted.

- If you are using a fact that we have not covered in class, please provide a proof for it. This applies even to facts that are published and well-known.

- If you are not able to solve a problem in full, make a simplifying assumption. Start early, do your best, and take pride in your discoveries.

Most importantly, have fun!

Midterm Exam

1 **ISOMORPHIC GRAPHS.** Alice and Bob each have an undirected graph on n vertices. How much communication do they need to find out deterministically if their graphs are isomorphic?

2 **INTEGER MULTIPLICATION.** How much deterministic communication does it take to compute the n^{th} least significant bit of the product of two natural numbers, of which Alice has one and Bob the other?

3 **BOOLEAN FORMULAS.** A *Boolean formula* in variables $z_1, \ldots, z_n$ is a fully parenthesized expression with operands $z_1, \neg z_1, \ldots, z_n, \neg z_n$ and operators $\wedge$ and $\vee$. Let $\varphi(z_1, \ldots, z_n)$ be a Boolean formula in which every variable occurs exactly once. Prove that computing $\varphi(x \oplus y)$ deterministically on input $x, y \in \{0,1\}^n$ requires $\Omega(n)$ bits of communication.

4 **JAZZY INNER PRODUCT.** Define $f\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ by $f(x,y) = 1$ iff $\sum x_i y_i \equiv 0 \pmod{18181}$. What is the nondeterministic communication complexity of f?

5 **RELATIVE PRIMALITY.** Alice and Bob's inputs are integers a and b, respectively, where $a, b \in [1, 2^n]$. Prove that $\Theta(n/\log n)$ bits are necessary and sufficient to verify nondeterministically that a and b are relatively prime.

6 **ORTHOGONAL SUBSPACES.** On input linear subspaces $A, B \subseteq \mathbb{F}_2^n$, prove that $\Theta(n^2)$ bits of nondeterministic communication are necessary and sufficient to check if A and B are orthogonal.

7 **COMMUNICATION VS. RANDOMNESS.** Prove that any randomized protocol for EQ_n with probability of correctness $2/3$ and communication cost c must use more than $\log_2(n/c)$ bits of randomness.

8 **BETTER THAN RANDOM.** Prove that every $f\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ has a randomized protocol with constant cost and error at most $\frac{1}{2} - \Theta(2^{-n/2})$.