University of California, Los Angeles
CS 289 Communication Complexity
Instructor: Alexander Sherstov

June 5, 2019

Instructions

- Your solutions are due at 11:59 pm on June 14, 2019. Please typeset your work and submit the resulting PDF document on CCLE using the TurnItIn feature. Late work will not be accepted or graded, resulting in zero credit for this assignment.

- You cannot collaborate with other students on this assignment. The work that you submit must be your own.

- This exam can be solved in its entirety using the course material taught so far, without consulting any additional sources. However, you are welcome to use any scholarly sources, including your textbook and the Internet. If you do, please write the solution in your own words and acknowledge the sources that you have consulted.

- If you are using a fact that we have not covered in class, please provide a proof for it. This applies even to facts that are published and well-known.

- If you are not able to solve a problem in full, make a simplifying assumption. Start early, do your best, and take pride in your discoveries.

Most importantly, have fun!

Final Exam

1 MULTILINEARITY. A polynomial in $x_1, x_2, \ldots, x_n$ is called *multilinear* if it is linear in each variable, e.g., $1,\ x_1 - x_2x_7,\ x_1x_2\cdots x_n$, but not x_1^2+1. Let $p \in \mathbb{F}_2[x_1, x_2, \ldots, x_n]$ be a multilinear polynomial of degree d. Prove that computing $p(x+y)$ deterministically on input $x, y \in \mathbb{F}_2^n$ requires $\Omega(d)$ bits of communication. How tight is this lower bound?

2 DISCREPANCY. Prove that $\operatorname{disc}(\mathrm{DISJ}_n) \leqslant O(1/\sqrt{n})$. *Hint:* there is a shockingly short proof that requires no calculations!

3 INNER PRODUCT. Prove that $R_{1/3}(\mathrm{IP}_n) \geqslant n - O(1)$.

4 INTERSECTION SIZE. If Alice and Bob are given sets $A, B \subseteq \{1, 2, \ldots, n\}$, respectively, how much communication do they need to deterministically compute $|A \cap B|$ to within $0.01n$?

5 HAMMING DISTANCE. Let PRIME_n be the problem of determining whether the Hamming distance between two given n-bit strings is a prime. Prove that $R_{1/3}(\mathrm{PRIME}_n) = \Omega(n)$.

6 SIGN FLIPS. Fix an arbitrary matrix $A \in \{-1, 1\}^{n\times n}$. A *flip* is an operation on A whereby one selects a row or column and flips the signs of all the entries in it. Let $\mathrm{fl}(A)$ denote the set of matrices that can be obtained from A by a finite sequence of flips. Prove the following:

a. $|\,\mathrm{fl}(A)| = 2^{2n-1}$;

b. $\mathrm{fl}(A)$ contains a matrix in which all row sums and all column sums are nonnegative;

c. $\mathrm{fl}(A)$ contains a matrix whose entries sum to $\Omega(n\sqrt{n})$;

d. the lower bound in **c.** is tight in general.

7 CONVEX RELAXATIONS. Use the approach of convex relaxations to derive the *generalized discrepancy method*: for any $\varphi \not\equiv 0$,

$$2^{R_\varepsilon(f)} \geqslant \frac{\langle (-1)^f, \varphi\rangle - 2\varepsilon\|\varphi\|_1}{\max_{\text{rectangle } R} |\langle R, \varphi\rangle|}.$$

8 POLYNOMIAL APPROXIMATION. Let $E(n,d) = \min_p \sum_{x\in\{0,1\}^n} |p(x_1, x_2, \ldots, x_n) - x_1x_2\cdots x_n|$, where the minimum is over all polynomials p of degree at most d. In words, $E(n,d)$ is the least ℓ_1 error in an approximation of the AND function by a polynomial of degree at most d. Determine $E(n,d)$ for $d = 0, 1, 2, \ldots, n$.

9 HAT COLOR. A group of n prisoners are brought into a room with n chairs arranged in a circle. Once they are all seated, each prisoner has a hat placed on their head. There are n distinct colors of hats, which are known to the prisoners beforehand, though colors may be repeated or not appear on any prisoner's head. Each prisoner can see everyone's hat color except for their own. The prisoners must then simultaneously call out a guess for their own hat color. The prisoners are set free if at least one of them guesses their own hat color correctly, and are all executed otherwise. Is it possible for the prisoners to guarantee that they will all be set free? The prisoners can devise a strategy before entering the room, but cannot communicate thereafter.