University of California, Los Angeles — December 10, 2014
CS 289 Communication Complexity
Instructor: Alexander Sherstov

Instructions

- Your solutions are due Friday, December 19, at 11:59pm. Please typeset your solutions and email me your PDF document. Late work will not be accepted or graded, resulting in zero credit for this assignment.

- You should think of this assignment as a take-home exam. Specifically, you cannot discuss this problem set with your fellow classmates or anyone else.

- In addition to the textbook and lecture notes, feel free to use any scholarly sources such as published research and Internet materials. Please remember to acknowledge any sources that you have consulted.

- If unable to solve a problem in full, try to derive a weaker bound or solve the problem under an unproven simplifying assumption.

Most importantly, have fun!

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Problem Set II

1 Prove that $\operatorname{disc}(\mathrm{DISJ}_n) \leqslant O(1/\sqrt{n})$. *Hint:* there is a very short proof that requires no calculations!

2 Prove that $R_{1/3}(\mathrm{IP}_n) \geqslant n - O(1)$.

3 Use the approach of convex relaxations to derive the *generalized discrepancy method*: for any $\varphi \not\equiv 0$,

$$2^{R_\varepsilon(f)} \geqslant \frac{\langle (-1)^f, \varphi\rangle - 2\varepsilon\|\varphi\|_1}{\max_{\text{rectangle } R} |\langle R, \varphi\rangle|}.$$

4 Prove that the choice of error parameter $0 < \varepsilon < 1/2$ affects the approximate degree of a Boolean function by at most a multiplicative constant, i.e., for any $\varepsilon, \delta \in (0, 1/2)$, we have $c_{\varepsilon,\delta} \deg_\varepsilon(f) \leqslant \deg_\delta(f) \leqslant C_{\varepsilon,\delta} \deg_\varepsilon(f)$ for some $C_{\varepsilon,\delta} > c_{\varepsilon,\delta} > 0$ and all Boolean functions f.

5 Let M be a nonnegative matrix. Define $\operatorname{rk}_+(M)$ to be the least k for which $M = M_1 + M_2 + \cdots + M_k$, where each M_i is a nonnegative matrix of rank 1. Assuming that $\operatorname{rk}_+(M) \leqslant \exp((\log_2 \operatorname{rk}_{\mathbb{R}} M)^c)$ for some constant $c > 1$ and every $M \in \{0,1\}^{n\times m}$, prove the log-rank conjecture.

6 A village has n residents. Every day at noon, they all meet at the main square to discuss daily matters. An evil spirit visits the village one night and marks the noses of k of the villagers with indelible ink, $k \geqslant 1$. Later that day, at noon, the evil spirit comes to the village meeting and announces that at least one villager has a marked nose. The evil spirit never visits the village again. If (and only if) a villager is able to logically deduce that his nose is marked, he will leave the village the same day never to be seen again. Nose marks being a taboo subject in town, the villagers never discuss it in any way. Moreover, a villager with a marked nose can never see his own mark. What will be the population count in the village $n + 1$ days after the evil spirit's visit?