University of California, Los Angeles   December 4, 2013
CS 289 Communication Complexity
Instructor: Alexander Sherstov

Instructions for Problem Set II

- Your solutions are December 12 at 11:59pm. Please typeset your solutions and email me your PDF document. Late work will not be accepted or graded, resulting in zero credit for this assignment.

- You cannot collaborate with other students on this assignment. The work that you submit must be your own.

- In addition to the textbook and lecture notes, feel free to use any outside sources such as the research literature and Internet materials. Remember to acknowledge any sources that you have consulted.

- If unable to solve a problem in full, try to derive a weaker bound or solve the problem under an unproven simplifying assumption.

Most important, have fun!

Problem Set II

1   Prove that $\mathrm{disc}(\mathrm{DISJ}_n) \leqslant 1/\sqrt{n}$.

2   Prove that $\mathsf{P} \subseteq \Sigma_1 \cap \Pi_1$.

3   Let $A \in \mathbb{R}^{n \times m}$. Prove that exactly one of the following systems of linear inequalities is feasible: (i) $Ax \geqslant 0$, $Ax \neq 0$;   (ii) $A^\mathsf{T} y = 0$, $y < 0$.

4   Given $f \colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, let f^k denote the problem of computing the tuple $(f(x_1, y_1), \ldots, f(x_k, y_k))$, where Alice gets $x_1, x_2, \ldots, x_k \in \{0,1\}^n$ and Bob gets $y_1, y_2, \ldots, y_k \in \{0,1\}^n$. Prove the strongest lower bound that you can on $D(f^k)$ in terms of $D(f)$.

5   Let M be a nonnegative matrix. Define $\mathrm{rk}_+(M)$ to be the least k for which $M = M_1 + M_2 + \cdots + M_k$, where each M_i is a nonnegative matrix of rank 1. Assuming that $\mathrm{rk}_+(M) \leqslant \exp((\log_2 \mathrm{rk}_\mathbb{R} M)^c)$ for some constant $c > 1$ and every $M \in \{0,1\}^{n \times m}$, prove the log-rank conjecture.

6   A village has n residents. Every day at noon, they all meet at the main square to discuss daily matters. An evil spirit visits the village one night and marks the noses of k of the villagers with indelible ink, $k \geqslant 1$. Later that day, at noon, the evil spirit comes to the village meeting and announces that at least one villager has a marked nose. The evil spirit never visits the village again. If (and only if) a villager is able to logically deduce that his nose is marked, he will leave the village the same day never to be seen again. Nose marks being a taboo subject in town, the villagers never discuss it in any way. Moreover, a villager with a marked nose can never see his own mark. What will be the population count in the village $n + 1$ days after the evil spirit's visit?