

Lecture #3

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1 Alternating Automata on Finite Words (AFW)

An alternating finite automaton (AFW) [1] is a non-deterministic finite automaton whose transitions are divided into existential and universal transitions. A run in an AFW can be seen as a game between the pathfinder, and the destroyer. At each turn, pathfinder and destroyer alternate between existential and universal clauses. The pathfinder wins the game if, after the input is completely consumed, the current state is an accepting state. In Figure 1 this game is represented as a tree.

Theorem 1. *For each AFW A , there is a NFA A' that recognizes the same language.*

Proof. Consider the following equalities:

- $AFW = (Q, \Sigma, \delta, q_0, F)$
- $NFW = (2^Q, \Sigma, \delta', \{q_0\}, 2^F)$
- $\delta'(S, a) = \{S' | S' \models \bigwedge_{q \in S} \delta(q, a)\}$

□

1.1 Closure Properties

Union

- $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$
- $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
- $A_1 \cup A_2 = (Q_1 \cup Q_2 \cup \{q_0\}, \Sigma, \delta', \{q_0\}, F_1 \cup F_2)$
- $\delta'(q_0, a) = \delta_1(q_{01}, a) \vee \delta_2(q_{02}, a)$

Intersection

- $\delta'(q_0, a) = \delta_1(q_{01}, a) \wedge \delta_2(q_{02}, a)$

Complementation The dual θ of $\theta \in B^+(Q)$ is defined as follows:

1. if $q \in Q$, $\bar{q} = q$
2. $\overline{true} = false \wedge \overline{false} = true$
3. $\overline{q_1 \vee q_2} = \bar{q}_1 \wedge \bar{q}_2$
4. The complement of $(Q, \Sigma, \delta, q, F)$ is $(Q, \Sigma, \bar{\delta}, q_0, Q \setminus F)$

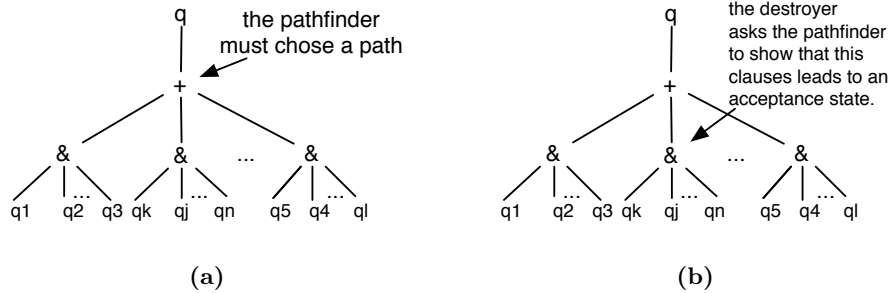


Fig. 1. (a) The pathfinder must chose one path among a set of disjunctions. (b) The destroyer attempts to find a non-accepting state among the conjunctions.

To test an AFW for emptiness is PSPACE-complete. The algorithm is as follows:

AFW \rightarrow NFW \rightarrow Check emptiness

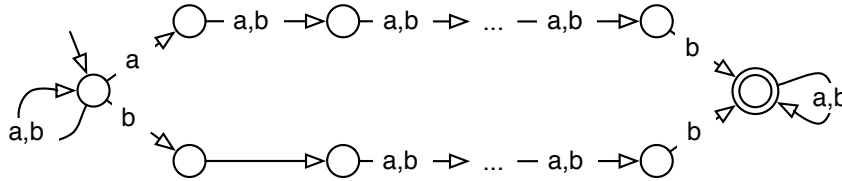
To test the AFW for universality is also PSPACE-complete:

AFW \rightarrow NFW \rightarrow complement \rightarrow Check emptiness

Example of Exponential Blow-up: Fix $n \geq 1$, let $\Sigma = \{a, b\}$:

$$L_n = \{uavbw \mid u, w \in \Sigma^*, v \in \Sigma^{n-1}\} \cup \{ubvaw \mid u, w \in \Sigma^*, v \in \Sigma^{n-1}\}$$

An AFW that recognizes L_n has $2n + 2$ states:



Any NFA that accepts $\overline{L_n}$ has an exponential number of states. To see this, suppose A is a NFW with $< 2^n$ states that accepts $\overline{L_n}$. A accepts all string of the form $ww, w \in \Sigma^n$. There must be two words $x, y \in \Sigma^*$ such that A is in the same state after reading x and y . But then A will have an accepting run on xy ; however, $xy \notin \overline{L_n}$

2 Automata on ω -words

Definition 1. An automaton on ω -word is a tuple $(Q, \Sigma, q_0, \delta, F)$, such that:

- Q is the set of states.
- Σ is the alphabet.
- q_0 is the Initial state.
- δ is a transition function: $\delta : Q \times \Sigma \rightarrow 2^Q$
- F is a set of final states.

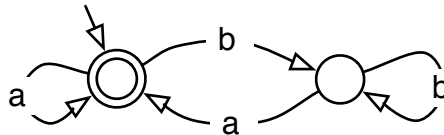
A run on a ω -word $\alpha = \alpha_0, \alpha_1, \dots$ is a sequence q_0, q_1, \dots of states of Q such that:

1. q_0 is the initial state.
2. $\forall i, r_i \rightarrow r_{i+1}, r_{i+1} \in \delta(r_i, \alpha_i)$

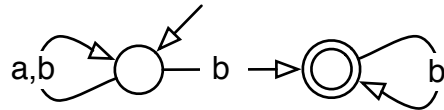
Automata on ω -words read infinite inputs. Consider a run ρ , and let $Inf(\rho) = \{r \in Q | r \text{ occurs } \infty\text{-ly many times in } \rho\}$. Run ρ is accepting if $Inf(\rho) \cap F \neq \emptyset$. This means that in an accepting run, some final state occurs ∞ -ly often. An ω -word α is accepted if it leads to an accepting run ρ .

L is a ω -regular language if there is a Büchi automaton A such that $L = L(A)$.
Ex.:

$$L_1 = \{\omega | \omega \text{ has infinitely many } a\text{'s}\}$$



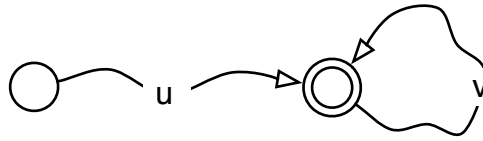
$$L_2 = \{\omega | \omega \text{ has finitely many } a\text{'s}\}$$



Theorem 2. *There is no DBW that accepts L_2 .*

Proof. Suppose A is a DBW that accepts L_2 . A accepts $\omega_i = b^\omega$. Suppose A visits a final state after n_1 steps. Now consider $\omega_i = b^{n_1} a b^\omega$. Let n_2 be the next occurrence of a final state: $\omega_i = b^{n_1} a b^{n_2} a b^\omega$. It is possible to continue with this argument to show one of two possibilities: (i) - A has an infinite number of final states, or (ii) - one of the final states of A is visited infinitely many times. Case (i) contradicts the definition of a finite automaton. Case (ii) would cause ω_1 to have infinitely many a 's. \square

Proposition 1. *If A is a Büchi Automaton and $L(A) \neq \emptyset$, then A accepts an eventually periodic word w^ω .*



2.1 Closure Properties

Union

- $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$
- $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
- $A_1 \cup A_2 = (Q_1 \cup Q_2 \cup \{q_0\}, \Sigma, \delta', \{q_0\}, F_1 \cup F_2)$
- $\delta'(q_0, a) = \delta_1(q_{01}, a) \vee \delta_2(q_{02}, a)$

Intersection

- $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$
- $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
- $A_1 \cap A_2 = (Q', \Sigma, \delta', q', F')$
 - $Q' = Q_1 \times Q_2 \times \{0, 1\}$
 - $q' = (q_{01}, q_{02}, 0)$
 - $\delta'((q_1, q_2, 0), a) = (q_{01}, q_{02}, 0)$

References

1. Ashok K. Chandra, Dexter C. Kozen, Larry J. Stockmeyer *Alternation*. Journal of the ACM, 28(1). 114-133, 1981.