CS 239 — Fall 2009	End Term #1
Cyber-Physical Systems	Due December 9, 2009

Problem 1. We start with an easy problem. Three periodic tasks A, B, and C are executed on a uniprocessor platform. The period of A is 6 units and worst case execution time is 0.5 units. The period of B is 12 units and worst case execution time is 1 unit. Suppose C has worst case execution time of 1 unit. What is the minimum period for C so that the system is still schedulable? (Assume the tasks are independent, and scheduling is pre-emptive.)

Problem 2. The decision problem for parity games takes a parity game and a state of the game, and returns "yes" iff the state is winning for player 0 in the parity game. Show that the decision problem for parity games is in NP $\cap$ coNP. To do this, proceed as follows. Guess a winning strategy of player 0, and show that once the strategy is fixed, you can check if it is winning for player 0 in polynomial time. Argue why guessing a strategy only requires a guess of size polynomial in the size of the game, and outline why the checking is polynomial time. You can use the result from Homework 2 that CTL model checking with strong fairness is in polynomial time. Finally, show that the winning condition of player 1 in a parity game is also a parity condition to show the problem is in coNP.

Problem 3. Consider the continuous control system

$$\dot{x} = f(x, u)$$

A state z is *reachable* from state x in time T if there exists a control u such that the trajectory with initial state x reaches the state z at time T, i.e., $z = \phi(x, T)$, where $\phi(x, t)$ is the trajectory of the system with initial state x. The *reachable set* from x in time T, written $R^T(x)$, is the set of all reachable states from x at time T. The reachable set from x, written $R(x)$, is the union, over all $T \geq 0$ of $R^T(x)$.

A state x is called an *equilibrium point* if there exists a control u such that $f(x, u) = 0$.

1. Show that $x \in R^T(x)$ for every equilibrium point x and every $T \geq 0$.

2. Let $x \in \mathbb{R}^n$, and let $\dot{x} = Ax + u$ be a linear system. Then 0 is an equilibrium point. Show that $R^T(0)$ is a subspace for each $T \geq 0$, that is, if $x_1 \in R^T(0)$ and $x_2 \in R^T(0)$, and $c_1, c_2 \in \mathbb{R}$, then $c_1 x_1 + c_2 x_2 \in R^T(0)$.

3. With the conditions of part (b), show that for each $\epsilon > 0$, we have $R(0) = R^\epsilon(0)$. That is, if some state is reachable from 0, it is reachable in arbitrarily small time. (Hint: use part (b), and the fact that every subspace has dimension at most n.)

Problem 4. For any set X, a binary relation $R \subseteq X \times X$ is *well founded* if there does not exist an infinite sequence $s_0, s_1, \ldots$ of elements from X such that for each $i \geq 0$, $(s_i, s_{i+1}) \in R$. For example, the relation $>$ over the natural numbers is well-founded, but the relation $>$ over the integers is not well-founded. A binary relation is *semi-well founded* if it is the finite union of well-founded relations. Show that a semi-well founded relation may not be a well-founded relation.

Let $S = (X, x_0, \rightarrow)$ be a system with set of states X, starting state $x_0 \in X$, and transition relation $\rightarrow$. Let R^+ denote the transitive closure of the transition relation, i.e., $(s, t) \in R^+$ iff there exist $s_0, s_1, \ldots, s_k$ such that $s_0 = s$, $s_k = t$, and for each $i \in \{0, \ldots, k-1\}$ we have $s_i \rightarrow s_{i+1}$. Let *Reach* be the set of reachable states of S.

We say S *terminates* if there is no infinite sequence of states

$$x_0 \rightarrow x_1 \rightarrow \ldots$$

Show that S terminates iff $R^+ \cap \mathit{Reach} \times \mathit{Reach}$ is semi-well founded. [Note: One direction of this result is somewhat hard, and requires a combinatorial result called Ramsey's theorem. Try it yourself first, and ask me for a hint if you get stuck.]

Problem 5. [Research question] As far as I know, this problem has not been studied so far. Consider a 2-player game (V_0, V_1, E), where each node $v \in V_0 \cup V_1$ is given by valuations to n bits (i.e., each state is a bitvector in $\{0,1\}^n$). Suppose that player 0 does not have full observation. Instead, player 0 can "measure" each bit in the state bitvector, and can get the "right" value of the bit with probability p. What is an appropriate notion of winning in these games? Can you design an algorithm that player 0 can use to win these games of incomplete information w.r.t. safety properties? What about parity properties?