An Unusual Numerical Relationship

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1 Introduction

For a right isosceles triangle with legs of unity, its hypotenuse length, $\sqrt{2}$, is related to aspects of its non-right angles, each of $45^\circ$.

2 Overview

Beginning with the following trigonometric relation:

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

S. Liapunov uses $x = 30^\circ$, $x + \frac{x}{2} = 45^\circ$, and $y = x/2$ to derive a statement that is an unusual truth about $\sqrt{2}$, namely:

$$\sqrt{2} = \frac{1 + \sqrt{x}}{\sqrt{x^2 + \sqrt{x}}}$$

Remarkably eq.(2) holds for any number $x$. (E.g., you may verify it for $x = 143$.)

3 Connections

The words above use the language of geometry. This note omits reasoning Liapunov sent me. It consists of just two pages and uses only simple algebra.

Students could benefit from geometric, trigonometric and algebraic exercises based on this topic. (They would need to apply skills usually taught in high school.
or a first year college mathematics course).

The more interesting question is how can testing eq.(2) numerically via a hand calculator get students to want to follow this kind of algebra?

What remains is the mystery: how did he think of this? Another question: is this a well-known fact or not?