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CONTINUOUS CONTROL WITH STOCHASTIC STOPPING TIME

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Abstract

The criterion for a continuous control process whose stopping time is a random variable of known cumulative probability distribution is expressed as a single improper integral. (Physical examples of interrupted control situations are: regulator component failure, catalyst deterioration, space vehicle midcourse guidance, and random waiting for a digital computer used in the control of several systems.) A necessary condition for an optimal trajectory is derived, shown to be time-invariant only for the exponential probability law, and related to reliability theory. The optimal feedback control law, optimal trajectory, and minimum expected cost are presented for time-invariant linear systems with quadratic cost and exponentially distributed stopping time. A simple condition on the feedback rule for time-varying linear systems with arbitrary stopping time probability distribution is derived. A connection with the deterministic theory is established via the limit as the variance approaches zero.

1. Introduction

In many control applications, the duration of the control process itself is uncertain—e.g., the time when the midcourse maneuver of a space vehicle will be accomplished is known, a priori, only approximately; a component may fail at a random time; or changes in the system to be controlled or our knowledge of the system may occur at times which are initially unknown. (Some other physical examples where such an interrupted control model seems useful are: catalyst deterioration in chemical plants, and random waiting for a digital computer used in the control of several systems.) Bellman and Kalaba [1], Zadeh [2], and Eaton [3] have considered interrupted control for discrete-time stochastic control systems. We consider a continuous deterministic control process where only the interruption time is stochastic and assume that the uncertainty regarding the duration of the process is summarized in a given cumulative probability distribution for the stopping time. Here the expected cost of control can be written as an improper integral involving the distribution and the integrand of the original (deterministic) criterion; in stochastic control, when the random variations in the system are independent of the stopping time, the improper integral provides a simplification of the corresponding expected cost criterion. For a scalar first-order system with quadratic criterion we use this formulation to derive a necessary condition for a control to be optimal in terms of the distribution of the stopping time. The condition is a second-order differential equation with coefficients that depend on the distribution only through the conditional rate of failure function of reliability theory. We illustrate its application by examining some simple cases in detail. The optimal control, minimum expected cost, and feedback rule are derived for a linear time-invariant system under the exponential probability law. For time-varying linear systems with arbitrary stopping time probability distribution the condition is shown equivalent to a simple type of Riccati equation in a linear feedback rule. Finally, the usual deterministic control problem is shown to be the limit of the interrupted control case as the variance of the stopping time distribution goes to zero.

2. Expected Cost Criterion

Let

$$J(\underline{u}) = \int_0^T g(t,\underline{x},\dot{\underline{x}},\underline{u},\dot{\underline{u}})dt \tag{1}$$

be the cost functional of a control process with state $\underline{x}$, control $\underline{u}$, and stopping time T, a random variable distributed as $F(\tau) = \text{Prob }[T \leqq \tau]$, where $F(0^-) = 0$ since $T \geqq 0$. (A dot above a function of one variable denotes the derivative.) We seek a control that minimizes the expected cost E(J), which we write as an improper Stieltjes integral:

$$E(J) = \int_0^\infty \int_0^T g(t,\underline{x},\dot{\underline{x}},\underline{u},\dot{\underline{u}})dt\ dF(\tau). \tag{2}$$

The following lemma reduces this to a single integral.

Lemma. If T is a random variable distributed as $F(\tau) = \text{Prob }[T \leqq \tau]$, $F(0^-) = 0$ $(T \geqq 0)$, then

$$E\left\{\int_0^T g(t,\underline{x},\underline{u})dt\right\} = \int_0^\infty g(t,\underline{x},\underline{u})[1-F(t)]dt. \tag{3}$$

Proof. For a given $\underline{u}$ and $T = \tau$, $\int_0^\tau g(t,\underline{x},\underline{u})dt = G(\tau)$ and $dG(\tau) = g(\tau,\underline{x}(\tau),\underline{u}(\tau))d\tau$. Hence the result follows from the definition of expectation and the integration by parts formula

$$\int_0^\infty G(\tau)dF = [G(\tau)F(\tau)]_0^\infty - \int_0^\infty F(\tau)dG(\tau).$$

3. A Necessary Condition

Let the system dynamics be described by the scalar first-order equation

$$\dot{x} = h(t,x,u). \tag{4}$$

Then a necessary condition can be found from (3) and (4) by an optimization procedure. For example, the Euler-Lagrange equations are

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0, \qquad \frac{\partial f}{\partial u} - \frac{d}{dt}\frac{\partial f}{\partial \dot{u}} = 0, \tag{5}$$

$$f(t,x,\dot{x},u,\dot{u}) = g(t,x,\dot{x},u,\dot{u})[1-F(t)] + \lambda(t)[\dot{x} - h(t,x,u)],$$

where $\lambda(t)$ is a Lagrange multiplier. Denoting partial differentiation with respect to a variable by writing it as a subscript and deleting explicit reference to the arguments of the functions g and h, we have

$$(g_x)(1-F) - \lambda h_x - \frac{d}{dt}[(g_{\dot{x}})(1-F)] - \frac{d\lambda}{dt} = 0, \tag{5a}$$

$$(g_u)(1-F) - \lambda h_u - \frac{d}{dt}[(g_{\dot{u}})(1-F)] = 0. \tag{5b}$$

If the system is of form $h = \ell(t,x) + bu$, $h_x = \ell_x$, and $h_u = b$. Then eliminating λ between (5a) and (5b) yields

$$(g_x)(1-F) - \frac{\ell_x}{b}\left[(g_u)(1-F) - \frac{d}{dt}[(g_{\dot{u}})(1-F)]\right] - \frac{d}{dt}[(g_{\dot{x}})(1-F)] - \frac{1}{b}\frac{d}{dt}[(g_u)(1-F)] + \frac{1}{b}\frac{d^2}{dt^2}[(g_{\dot{u}})(1-F)] = 0. \tag{6}$$

Thus Eqs. (4) and (6) together are a necessary condition for an optimal trajectory for minimum expected cost under stochastic stopping with probability distribution F. Letting b = 1 and simplifying yields

$$\dot{x} = \ell(t,x) + u, \tag{4'}$$

$$\frac{d^2}{dt^2}\,[(g_{\dot u})(1-F)] + \ell_x \frac{d}{dt}\,[(g_{\dot u})(1-F)] \qquad (6')$$

$$-\frac{d}{dt}\,[(g_{\dot x}+g_u)(1-F)] + (g_x-\ell_x g_u)(1-F) = 0.$$

The extension of this approach to the vector case is straightforward. Hence the details will be omitted. The vector results are summarized in Sec. 7. In what follows we present a detailed scalar example.

4. Quadratic Cost

When g is known analytically, (6') can be simplified and combined with (4') to yield a single differential equation in the state or control whose solution yields the optimal trajectory. For example, let $g = x^2 + u^2$. Then (6') becomes

$$-\frac{d}{dt}\,[u(1-F)] + (x-\ell_x u)(1-F) = 0 \qquad (7)$$

When Prob $[T \le t] < 1$, this can be written as

$$x = \ell_x u + \frac{1}{1-F}\frac{d}{dt}\,[u(1-F)]. \qquad (8)$$

Eliminating u between (4') and (8):

$$x = \ell_x(\dot x-\ell) + \frac{1}{1-F}\frac{d}{dt}\,[(\dot x-\ell)(1-F)] \qquad (9)$$

For a continuous probability law of density $f = \dot F$, this is

$$\ddot x - \frac{\dot F}{1-F}\,\dot x - \ell\,[\ell_x - \frac{\dot F}{1-F}] - \ell_t - x = 0, \qquad (10)$$

a second-order nonlinear differential equation which depends on the stopping time distribution only through the ratio of density to one minus cumulative distribution.

5. Stopping Time Distributions

The only continuous probability law for which (10) could be time-invariant is the exponential law. It is necessary for time-invariance that

$$\frac{-\dot F}{1-F} = \frac{d(1-F)}{(1-F)} = -\nu, \qquad (11)$$

where ν is a constant. This implies $\ln[1-F(t)] = -\nu t$. Thus

$$F(t) = \text{Prob}\ (T \le t) = 1 - e^{-\nu t}, \quad t \ge 0, \qquad (12)$$

the exponential law with mean = E(T) = $1/\nu$. If the system is autonomous, i.e., if h does not depend on time explicitly, $\ell(t,x) = \ell(x)$. Then for this probability distribution (10) becomes

$$\ddot x - \nu\dot x - \ell(x)[\dot\ell(x)-\nu] - x = 0, \qquad (13)$$

a nonlinear equation with constant coefficients.

In general, if we put $\nu = \nu(t)$ Eq. (11) shows that $\nu(t)$ completely characterizes a probability law. The distribution and density are then easily found to be

$$F(t) = 1 - e^{-\int_0^t \nu(\tau)d\tau} \quad \text{and}$$

$$f(t) = \dot F(t) = \nu(t)\, e^{-\int_0^t \nu(\tau)d\tau}$$

The ratio of density to one minus cumulative distribution, $\nu(t)$, is called "the conditional rate of failure" function in reliability theory, since $\nu(t)dt$ = Prob $[t < T \le t + dt | T > t]$. A table of this function for several probability laws is presented below.

6. Time-Invariant Linear Systems

When the system dynamics are linear, Eq. (13) can be solved analytically. Furthermore, Lee [4] states that the three conditions

1. $[(g)(1-F)]$ strictly convex with respect to $\underline{u}$,
2. $[(g)(1-F)]$ convex with respect to $\underline{x}$,

and

3. h linear in $\underline{x}$ and $\underline{u}$,

which are satisfied in this case, are sufficient for the solutions of the Euler equations to be the unique global minimum. Hence the optimal control will be found in this case as follows.

Put $\ell(x) = ax$. Then

$$\dot{x} = ax + u \tag{4''}$$

and

$$\ddot{x} - \nu\dot{x} - [1 + a^2 - a\nu]x = 0 \tag{14}$$

The roots of the characteristic equation of (14) are

$$\frac{\nu}{2} \pm \sqrt{(\frac{\nu}{2})^2 + 1 + a^2 - a\nu} \tag{15}$$

$$= \frac{\nu}{2} \pm \sqrt{1 + (a - \frac{\nu}{2})^2} = \frac{\nu}{2} \pm R.$$

Then, applying the initial condition $x(0) = c$,

$$x = \alpha\, e^{(\frac{\nu}{2} + R)t} + (c-\alpha)\, e^{(\frac{\nu}{2} - R)t}, \tag{16}$$

$$u = \alpha(\frac{\nu}{2} + R - a)\, e^{(\frac{\nu}{2} + R)t} \tag{17}$$

$$+ (c - \alpha)(\frac{\nu}{2} - R - a)\, e^{(\frac{\nu}{2} - R)t}.$$

This implies

$$E(J) = \int_0^\infty (x^2 + u^2)\, e^{-\nu t}\, dt \tag{18}$$

$$= \int_0^\infty \Big\{\alpha^2\Big[1 + (\frac{\nu}{2} + R - a)^2\Big]\, e^{(\nu+2R)t}$$

$$+ 2\alpha(c-\alpha)\Big[1 + (\frac{\nu}{2} - a)^2 - R^2\Big]\, e^{\nu t}$$

$$+ (c-\alpha)^2\Big[1 + (\frac{\nu}{2} - R - a)^2\Big] e^{(\nu-2R)t}\Big\}\, e^{-\nu t}\, dt$$

$$= \Big[\alpha^2(R + \frac{\nu}{2} - a)\, e^{2Rt} - (c-\alpha)^2(R - \frac{\nu}{2} + a) e^{-2Rt}\Big]_0^\infty.$$

Therefore α must be zero to keep the expected cost of control finite.

Hence a linear system with initial state $x(0) = c$, quadratic cost, and exponentially distributed stopping time has an optimal trajectory

$$x^o(t) = c\, e^{(\frac{\nu}{2} - R)t}, \tag{19}$$

the corresponding feedback optimal control (from (19) and (4''))

$$u^o(t) = -(a - \frac{\nu}{2} + R)x^o(t), \tag{20}$$

and a minimum expected cost of control

$$E(J^o) = c^2(a - \frac{\nu}{2} + R). \tag{21}$$

As $\nu \to 0$, so that the expected terminal time $E(T) = \overline{T} = \frac{1}{\nu} \to \infty$, the optimal control, trajectory, and criterion become the corresponding optimal quantities for the regulator (infinite-duration) problem with linear dynamics and quadratic cost.

7. Time-Varying Linear Systems

If the stopping time is not exponentially distributed, for a linear time-varying system $\dot{x} = a(t)x + u$, Eq. (10) becomes

$$\ddot{x} - \frac{\dot{F}}{1-F}\, \dot{x} - [1 + a^2(t) + \dot{a}(t) - \frac{\dot{F}}{1-F}\, a(t)]x = 0. \tag{22}$$

The change of variable

$$w = \frac{\dot{x}}{x} \tag{23}$$

shows that this is equivalent to the Riccati equation

$$\dot{w} + w^2 - \frac{\dot{F}}{1-F}\, w - [1 + a^2(t) + \dot{a}(t) - \frac{\dot{F}}{1-F}\, a(t)] = 0. \tag{24}$$

Equation (24) may be simplified and given a physical interpretation. Define the linear feedback rule G(t) by

$$u(t) = G(t)x(t). \quad (25)$$

Then by (4") and (23)

$$w = a(t) + G(t). \quad (26)$$

Substitution of (26) in (24) leads to the following condition on $G(t)$:

$$\dot{G} + G^2 + [2a(t) - \frac{\dot{F}}{1-F}]G - 1 = 0. \quad (27)$$

Equation (27) is a simple necessary and sufficient condition on the linear feedback rule for minimum expected cost with quadratic criterion. For specific $a(t)$ and $F(t)$, solutions can be obtained by several methods. If $a(t)$ and $F(t)$ cause (27) to have the coefficient of G a constant, say Q, (as for $F(t)$ the uniform law on $[0,M]$, $a(t) = \frac{1}{2(M-t)} + Q$) a one parameter family of solutions is

$$G(t) = \frac{-[\frac{Q}{2} + R_Q] + k[\frac{Q}{2} - R_Q]e^{-2R_Q t}}{1 - k\, e^{-2R_Q t}} \quad (28)$$

where

$$R_Q = \sqrt{(\frac{Q}{2})^2 + 1}\ .$$

(The solution of the previous section is Eq. (28) with $k = 0$ the appropriate parameter.) If $a(t) = a$ and F is Rayleigh with $\alpha = 1$

$$G(t) = \frac{e^{-(\frac{t^2}{2} - 2at)}}{k + \int_0^t e^{-(\frac{\tau^2}{2} - 2a\tau)} dt} + t - 2a. \quad (29)$$

(The solution also holds for $a(t) = -t/2$ and F exponential with parameter ν if we replace "2a" by "ν" in (29).)

The analogs of (22) and (27) can be derived for the vector case. Let $\underline{x}$ and $\underline{u}$ be column vectors; prime denote transpose; and $A = A(t)$, P, Q, $G(t)$ and the identity I, be matrices of appropriate order. Then for

$$\dot{\underline{x}} = A\underline{x} + \underline{u}, \quad (30)$$

when we seek a trajectory $\underline{x}$ or control $\underline{u}$ which minimizes

$$J = \int_0^T (\|\underline{x}\|_P^2 + \|\underline{u}\|_Q^2) dt = \int_0^T (\underline{x}'P\underline{x} + \underline{u}'Q\underline{u}) dt, \quad (31)$$

we have

$$\ddot{\underline{x}} + \left[Q^{-1}A'Q - A - I\,\frac{\dot{F}}{1-F}\right]\dot{\underline{x}} \quad (32)$$
$$- \left[Q^{-1}(P + A'QA) + Q\dot{A} - A\,\frac{\dot{F}}{1-F}\right]\underline{x} = \underline{0}$$

or

$$\dot{G} + G^2 + \left[GA + Q^{-1}A'QG - G\,\frac{\dot{F}}{1-F}\right] \quad (33)$$
$$-Q^{-1}P = 0$$

as necessary and sufficient optimality conditions when P is nonnegative definite and Q is positive definite.

8. Limit as Variance Approaches Zero

A relationship between control with stochastic stopping-time and the usual deterministic case is given by the following theorem.

Theorem. Let $F_\sigma(t)$ be a probability distribution on $[0,\infty)$ of mean $\bar{T} < \infty$ and variance $\sigma^2 < \infty$, $g(t) = g(t, \underline{x}(t), \underline{u}(t))$ be nonnegative and such that

$\int_0^\infty \frac{g(t)}{1+t^2} dt < \infty$ for all $\underline{x}(t)$, $\underline{u}(t)$, and $t \in [0,\infty)$. Then

$$\lim_{\sigma^2 \to 0} E\left\{\int_0^T g(t)dt \,\middle|\, T \sim F_\sigma(t)\right\} = \int_0^{\bar{T}} g(t)dt. \quad (34)$$

Proof. The lemma yields the equivalent conclusion:

$$\lim_{\sigma^2\to 0} \int_0^\infty g(t) \cdot [1-F_\sigma(t)]dt = \int_0^{\bar{T}} g(t)dt. \tag{35}$$

Since

$$\int_0^{\bar{T}} g(t)dt = \int_0^\infty g(t) \cdot [1-F(t)]dt, \tag{36}$$

where

$$F(t) = \lim_{\sigma^2\to 0} F_\sigma(t) = \begin{cases} 0, & t < \bar{T} \\ 1, & t \geqq \bar{T}, \end{cases} \tag{37}$$

this will be established by dominated convergence. The inequality (38) will be used

$$1-F_\sigma(t) \leqq \frac{\sigma^2 + \bar{T}^2}{t^2} \tag{38}$$

$$\Big[\sigma^2 = E\{T^2\} - E^2\{T\} = \int_0^\infty s^2\, dF_\sigma(s) - \bar{T}^2$$

$$\therefore\ \sigma^2 + \bar{T}^2 = \int_0^\infty s^2\, dF_\sigma(s) \geqq \int_t^\infty s^2\, dF_\sigma(s)$$

$$\geqq t^2 \int_t^\infty dF_\sigma(s) = t^2[1-F_\sigma(t)]\Big].$$

Define $r^2 = \sigma^2 + \bar{T}^2$. Then, since $[1-F_\sigma(t)] \leqq 1$,

$$g(t) \cdot [1-F_\sigma(t)] \leqq \Gamma(t) = \begin{cases} g(t), & t < r \\ g(t)\cdot r^2/t^2, & t \geqq r \end{cases} \tag{39}$$

$\Gamma(t)$ is summable, i.e.,

$$\infty > \int_0^\infty \Gamma(t)dt = \int_0^r g(t)dt + \int_r^\infty \frac{g(t)}{t^2}\cdot r^2 dt, \tag{40}$$

since the hypotheses $g(t)$ nonnegative and $\int_0^\infty \frac{g(t)}{1+t^2}\, dt < \infty$, imply $\int_0^r \frac{g(t)}{1+t^2}\, dt < \infty$, and $\int_r^\infty \frac{g(t)}{1+t^2}\, dt < \infty$, and since

$$\int_0^r \frac{g(t)}{1+t^2}\, dt < \infty \Rightarrow \int_0^r g(t)\, dt < \infty \tag{41}$$

and

$$\int_r^\infty \frac{g(t)}{1+t^2}\, dt < \infty \Rightarrow \int_r^\infty \frac{g(t)}{t^2} \cdot r^2\, dt < \infty\ .* \tag{42}$$

Hence Lebesque's theorem on dominated convergence (see e.g., [5] Natanson, p. 161) applies and yields (2).

9. Conclusion

We have shown that it is possible to find optimal controls and feedback rules for continuous control systems with stochastic stopping time. Results for the case of a scalar system with quadratic criterion have been given to illustrate a procedure for incorporating the possibility of interruption into the analysis of a control process. A connection with the deterministic theory has been established by allowing the variance of the stopping-time distribution to approach zero.

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$$\infty > \int_0^r \frac{g(t)}{1+t^2}\, dt \geqq \frac{1}{1+r^2} \int_0^r g(t)\, dt \Rightarrow \tag{41}$$

$$\infty > \int_0^r g(t)\, dt$$

$$\infty > \int_r^\infty \frac{g(t)}{1+t^2}\, dt \Rightarrow \infty > \int_r^\infty \frac{g(t)}{t^2}\, dt \tag{42}$$

since for any r there always is a finite constant k such that $\frac{g(t)}{1+t^2} > k\, \frac{g(t)}{t^2}$.

(The converse implications also hold.)

References

1. Bellman, R., and R. Kalaba, "A Note on Interrupted Stochastic Control Processes," Information and Control Vol. 4, No. 4, 1961, pp. 346–349.

2. Zadeh, L. A., "Remark on the Paper by Bellman and Kalaba," Information and Control, Vol. 4, No. 4, 1961, pp.350–352.

3. Eaton, J. H., "Discrete-Time Interrupted Stochastic Control Processes," J. Math. Anal. Appl., Vol. 5, No. 2 1962, pp. 287–305.

4. Lee, I., "Optimal Trajectory, Guidance, and Conjugate Points," Information and Control, Vol. 8, No. 6, 1965, pp. 589–606.

5. Natanson, I. P., Theory of Functions of a Real Variable, Vol. I, Frederick Ungar Publishing Co., New York, 1961.

RATIO OF DENSITY TO ONE MINUS CUMULATIVE DISTRIBUTION FOR SOME CONTINUOUS PROBABILITY LAWS

PROBABILITY LAW	DENSITY $\dot{F} = f(t)$ FOR $t \geq 0$ ($f(t) = 0,\ t < 0$)	RATIO $\dot{F}/(1 - F)$	MEAN	VARIANCE
Exponential, $\nu > 0$ [Gamma, $r = 1$]	$f(t) = \nu e^{-\nu t}$	ν	$\frac{1}{\nu}$	$\frac{1}{\nu^2}$
Rayleigh, $\alpha > 0$ [Chi, $n = 2$, $\sigma = \alpha\sqrt{2}$]	$f(t) = \frac{1}{\alpha^2}\, t\, e^{-\frac{1}{2}(\frac{t}{\alpha})^2}$	$\frac{t}{\alpha^2}$	$\alpha\sqrt{\frac{\pi}{2}}$	$\alpha^2[2 - \frac{\pi}{2}]$
Uniform, $M > 0$	$f(t) = \frac{1}{M},\ t \in [0, M]$	$\frac{1}{M - t},\ t \in [0, M)$	$\frac{M}{2}$	$\frac{M^2}{12}$
Pareto, $\alpha > 1$	$f(t) = \frac{(\alpha - 1)}{(1 + t)^\alpha}$	$\frac{\alpha - 1}{1 + t}$	$\frac{1}{\alpha - 2},\ \alpha > 2$	$\frac{\alpha - 1}{(\alpha - 3)(\alpha - 2)^2},\ \alpha > 3$
Gamma, $r = 1, 2, \ldots,\ \nu > 0$ [Chi-Square, $n = 2r$, $\sigma = \sqrt{2\nu}$]	$f(t) = \frac{\nu}{(r - 1)!}(\nu t)^{r-1} e^{-\nu t}$	$\frac{\nu(\nu t)^{r-1}/(r - 1)!}{\sum_{i=0}^{r-1} (\nu t)^i/i!}$	$\frac{r}{\nu}$	$\frac{r}{\nu^2}$
Chi, $n = 1, 2, \ldots,\ \sigma > 0$	$f(t) = \frac{\left[\frac{t}{\sigma}\right]^{n-1}\left[\frac{n}{2}\right]^{n/2}\left[\frac{2}{\sigma}\right]e^{-\frac{n}{2}(\frac{t}{\sigma})^2}}{\Gamma\left[\frac{n}{2}\right]}$	$\frac{\left[\frac{t}{\sigma}\right]^{n-1}\left[\frac{n}{2}\right]^{n/2}\left[\frac{2}{\sigma}\right]}{\Gamma\left[\frac{n}{2}\right]\sum_{i=0}^{n/2-1}\left[\frac{t}{\sigma}\right]^{2i}\left[\frac{n}{2}\right]^{i}\frac{1}{i!}}$, for $n = 2, 4, 6, \ldots$	$\frac{\sigma}{2^{n-1}}\sqrt{\frac{2\pi}{n}}\,\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}$	$\sigma^2\left[1 - \frac{\pi}{2^{2n-3}}\frac{\Gamma^2(n)}{\Gamma^2(\frac{n}{2})}\right]$
Weibull, $\lambda > 0$, $\alpha > 0$ [$\alpha = 1$ yields Exponential; $\alpha = 2$ yields Rayleigh]	$f(t) = \lambda\alpha t^{\alpha-1} e^{-\lambda t^\alpha}$	$\lambda\alpha t^{\alpha-1}$	$\Gamma(\frac{1}{\alpha} + 1)\,\lambda^{-\frac{1}{\alpha}}$	$\lambda^{-\frac{2}{\alpha}}\left\{\Gamma(\frac{2}{\alpha}+1) - \left[\Gamma(\frac{1}{\alpha}+1)\right]^2\right\}$
Modified Extreme Value, $\lambda > 0$	$f(t) = \frac{e^{t - \frac{e^t - 1}{\lambda}}}{\lambda}$	$\frac{e^t}{\lambda}$		