

(Laplace transform) transfer functions from discrete-time data at special times.

In many cases, however, the independent variable in an identification experiment is neither regularly spaced nor under the control of the experimenter. For example, periodic data may be obtained with gaps—as when a communications link to a spacecraft is lost temporarily. The purpose of this paper is to extend one of the above identification procedures to the situation where the experimenter cannot choose when observations can be obtained, or cannot rely on their being periodic, in a way that reduces interpolation inaccuracies, and the effect of measurement errors and noise.

The transfer function of a linear, lumped-parameter, time-invariant, single-input ($f(t)$), single-output ($x(t)$) system expressed as the ratio of polynomials with constant coefficients:

$$W(s) = \frac{N(s)}{D(s)} = \frac{b_\mu s^\mu + b_{\mu-1} s^{\mu-1} + \cdots + b_0}{a_\nu s^\nu + a_{\nu-1} s^{\nu-1} + \cdots + a_0}, \quad (1)$$

($\mu \leq \nu$ below and in most systems) leads to the transform-domain input-output-state relation:

$$X(s) = \left[\sum_{i=1}^{\nu} s_i(0^-) s^{\nu-i} + N(s)F(s) \right] / D(s), \quad (2)$$

when the effect of the initial state $s(0) = [s_i(0^-)]'$ is included. (See [6], ff. 227, for equivalent system representations.) Identification of the system is the estimation of $\mu, \nu, b_0, \dots, b_\mu, a_0, \dots, a_{\nu-1}$ (we put $a_\nu = 1$) and the initial state. In what follows we assume that μ and ν are known and examine the determination of the remaining ($2\nu + \mu + 1$) parameters.

Reference [5] suggests that the initial state and the transfer function parameters can be found by making the theoretical output transform of (2) agree with approximations to the transform derived from measurements of $x(t)$ and $f(t)$ by least squares. A quadratic in the unknowns,

$$S = \sum_s \left\{ \hat{X}(s) \sum_{i=0}^{\nu} a_i s^i - \hat{F}(s) \sum_{j=0}^{\mu} b_j s^j - \sum_{k=1}^{\nu} s_k(0^-) s^{\nu-k} \right\}^2 \quad (3)$$

is minimized by the solution to the set of $(2\nu + \mu + 1)$ linear algebraic equations in the $(2\nu + \mu + 1)$ parameters:

$$\frac{\partial S}{\partial s_i(0^-)} = \frac{\partial S}{\partial a_{i-1}} = \frac{\partial S}{\partial b_j} = 0, \quad \begin{aligned} i &= 1, 2, \dots, \nu, \\ j &= 0, 1, \dots, \mu. \end{aligned} \quad (4)$$

The sum in (3) is on real positive s , for numerical convenience. In most cases, summation produces differences in the coefficients of the unknowns in (4) which make these equations independent. If the input can be chosen by the experimenter, $\hat{F}(s) = F(s)$ should be other than 0, 1, $s^{-1}, \dots, s^{-\mu}$

(e.g., $F(s) = 1 + s^{-1}$), since for these values (4) may be dependent.

One way to incorporate aperiodic discrete-time data is to use the integrated Lagrange-interpolation relation:

$$\begin{aligned} \hat{X}(s) &= \mathcal{L}[T'V^{-1}Y] = M'(s)V^{-1}Y, \\ Y' &= [x(t_1), \dots, x(t_n)], \\ T' &= [1, t_1, \dots, t_1^{n-1}], \\ M'(s) &= \mathcal{L}[T'] \\ &= \left[\frac{1}{s}, \frac{1}{s^2}, \frac{2!}{s^3}, \dots, \frac{(n-1)!}{s^n} \right], \\ V &= \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix} = [t_i^{j-1}]. \end{aligned} \quad (5)$$

Here the $n \times n$ Vandermonde matrix V depends on the actual observation times t_i , and is nonsingular for distinct t_i . The above matrix representation of interpolation and several results used below are given in [10]. Equation (5) is also the numerical analysis "method of moments" for obtaining quadrature weights [7]. When the discrete-time observations are inexact, the least-squares regression polynomial of degree $(m-1)$ fitted to n data can be integrated. Then, with V_{nm} denoting the n -row, m -column matrix corresponding to V , the transform of the polynomial found from the normal equations is

$$\begin{aligned} \hat{X}(s) &= \mathcal{L}[T'(V_{nm}'V_{nm})^{-1}V_{nm}Y] \\ &= M'(s)(V_{nm}'V_{nm})^{-1}V_{nm}Y. \end{aligned} \quad (6)$$

$V_{nm}'V_{nm}$ is related to the ill-conditioned Hilbert matrix when the t_i are uniformly distributed, so the numerical matrix inversion can be difficult for large m . Rice [10] states that this has little effect for $m=4$ or 5.

These approximate transform expressions can be modified to account for non-polynomial integrands ($x(t)$). For example, other functions—exponentials, sinusoids, damped sinusoids—(and their \mathcal{L} -transforms) can replace the powers (and transforms of powers) of t in V and T' (and $M(s)$) of (5). In [5], a change of variable used to adapt a stored-weight formula to exponential polynomials; Reference [9] discusses formulas for the approximate integration of periodic functions; the numerical results below were for interpolation by exponentials: $V = [\exp(-(j-1)t_i)]$, $M'(s) = [1/(s+j-1)]$. In this case the procedure extends Prony's method [7] to aperiodic, noisy data (using (6)).

The sensitivity to noise of the above procedure is, as the numerical examples below indicate, appreciable. (Reference [8] discusses one aspect of this.)

Aperiodic discrete-time data from

$$\begin{aligned} x(t) &= 2 \exp(-t) + \exp(-2t) \\ &= c_1 \exp \alpha_1 t + c_2 \exp \alpha_2 t \end{aligned} \quad (7)$$

were used to investigate the application of (4) and (5) to the system described by the differential equation

$$\begin{aligned} \frac{d^2x}{dt^2} + 3 \frac{dx}{dt} + 2x &= 0, \quad x(0) = 3, \\ \frac{dx}{dt}(0) &= -4. \end{aligned} \quad (8)$$

Identification from Aperiodic Discrete-Time Data and Estimation of Exponential Parameters

Several authors have proposed system identification methods in which discrete-time data are used. Kalman [1], Lendaris [2], Levin [3], and Steiglitz and McBride [4], use periodic sampled data to estimate pulse transfer function coefficients. Bellman, Kagiwada, and Kalaba [5] discuss the estimation of parameters of conventional

TABLE I
ADDITIVE NOISE AND EFFECT OF OVERDETERMINATION

Equations	σ	a_1	a_0	c	c_1	α_1	c_2	α_2
(5)	10^{-5}	2.64	1.57	-3.96	1.49	-0.911	1.51	-1.73
(6)	10^{-3}	2.78	1.74	-3.98	1.72	-0.949	1.28	-1.83

TABLE II
EFFECT OF INTERPOLATION INACCURACIES

Experiment		a_1	a_0	c	c_1	α_1	c_2	α_2
I:	True	3.02	2.03	-4.03	2.00	-1.01	1.00	-2.01
	Est.	3.10	2.13	-4.04	2.08	-1.02	0.92	-2.08
II:	True	3.20	2.31	-4.30	2.00	-1.10	1.00	-2.10
	Est.	3.86	3.16	-4.37	2.45	-1.18	0.555	-2.69

The parameters sought were $a_1=3$, $a_0=2$, and $c=x(0)=-4$; $x(0)=3$ was assumed known. Once a_1 , a_0 and c were found, we calculated the corresponding c_1 , c_2 , α_1 , and α_2 . Exact observations $x(t_i)$ at $t_i=0.1, 0.2, 0.3, 0.4, 0.6, 0.7, 1.2, 1.3, 2.0, 2.1, 3.6$, and 3.7 were used; calculations were done on an IBM 7044 computer; matrix inversion was done by a standard routine. When double precision was used, exact estimates of the parameters were found using all the data and using as few as five observations. In single precision, using only five observations, we found $a_1=3.01$, $a_0=2.015$, $c=-4.00$ ($c_1=2.01$, $\alpha_1=-1.00$, $c_2=0.986$, $\alpha_2=-2.01$). Similar results would have been obtained had we used the interpolation of (5) to compute values of $x(t)$ at the times of the stored-weight formula used in [5]. (These times lie between pairs of adjacent t_i separated by 0.1 in the above set.) However, linear interpolation gave $a_1=2.05$, $a_0=0.865$, $c=-3.01$ ($c_1=0.531$, $\alpha_1=-0.594$,

$c_2=2.47$, $\alpha_2=-1.46$). By way of further comparison we used linear interpolation to find an ordinate at $t=0.5$ and then tried to use Prony's method on the equally spaced set from $t=0(0.1)0.7$. The calculation gave the meaningless result, $\exp(\alpha_2 t)$ =an imaginary number.

The effect of inaccurate observations was examined as follows: Zero-mean Gaussian noise was added to the twelve $x(t_i)$; estimates were obtained using (5)—interpolation—and also using (6)—overdetermination—for $m=n/2=6$. For interpolation, a standard deviation of 10^{-5} caused the inaccurate parameter estimates in Table I; with overdetermination, the comparable inaccuracies in the table were produced at a 10^{-3} standard deviation.

Noiseless data generated by the true parameters in Table II led to the corresponding estimates when interpolation was by an exponential polynomial (powers of $\exp(-t)$).

The above results suggest that in the

general case the following techniques should be used:

- Overdetermination with $n \gg m$ to suppress noise.
- Reservation of data at the extremes in time to check the estimates by extrapolation to reduce estimate errors caused by an inappropriate interpolation set.
- Iteration (using the last estimates to choose the new interpolation functions) to improve the accuracy of the estimates.

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