ALLEN KLINGER
THOMAS A. BROWN

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# Allocating Unreliable Units to Random Demands

1. Introduction. This paper is concerned with the allocation of a fixed inventory of unreliable units to a random number of demands. Qualitatively, only one unit of those allocated has to satisfy a demand. However, each unit will only satisfy a demand with a given probability; in that sense the units are "unreliable". The goal of an allocation strategy is to meet all demands encountered, that is, to have at least one allotted unit satisfying each demand. Two other measures of success will be discussed below: the expected number of consecutive demands met and the expected inventory remaining after meeting a sequence of demands successfully.

The model we deal with can be applied to situations where

(1) all the unreliable units have the same probability $p$ of functioning successfully, where $0 < p < 1$;

(2) demands for units occur at random times but only one demand occurs in any infinitesimal small time-interval;

(3) the random events of "a unit functioning" and "a demand occurring" are independent of each other; and

(4) allocated units are not reusable.

This model was originally developed for an operations research analysis of a military system. However, it is also well suited to (a) an inventory allocation with random customer arrivals where $p$ represents the probability that a given good will satisfy a customer and (b) an allocation of communication channels to messages which arrive at random when use of several parallel channels increases the overall probability of reliable transmission of a message. (Here, the assumption that the channels are not reusable corresponds to single message transmission times being much larger than the expected interval between

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successive messages.)

The original application considered a bomber which had $t$ uni of time remaining to accomplish its mission, $n$ units of air-to-air or bomber-defense missiles on board and expected to meet enemy interceptors generated by a nonstationary Poisson process with known statistics. The $n$ units were defensive armament to be used against the enemy interceptors; each had the same "kill probability". We origina assumed that an interceptor which had not been shot down was certain to destroy the bomber. This was relaxed in the analysis of other mode to include interceptors with kill probability less than one. In the following we refer to such a situation as an "unreliable demand", by ana ogy with the unreliable units (missiles), and speak of probabilities that the demand is sustained or withdrawn rather than kill probability. The presentations of both models are substantially independent.

The overall purpose of the initial model is best given by the title of the classified RAND report on the subject, "Calculating the Va of Bomber Defense Missiles", [1]. In addition, the functions defined below give the strategy (firing doctrine) and maximum probability of success, i.e. reaching destination having satisfied all demands (destroying all enemy interceptors) enroute.

## 2. The Basic Model.

2.1 Allocation to Poisson Demands. Let $t$ represent the am of time remaining during which demands may be encountered. We ass that there exists a continuous positive function $r(t)$ such that for an small interval of time $\Delta t$

(i) the probability that exactly one demand occurs in the inte val $(t-\Delta t/2,\ t+\Delta t/2)$ is $r(t)\Delta t + o(\Delta t)$†;

(ii) the probability that exactly zero demands occur in the int val is $1 - r(t)\Delta t - o(\Delta t)$; and

(iii) the probability that two or more events occur in the inte is $o(\Delta t)$. These assumptions are those of the Poisson probability la Define $m(t) \equiv \int_0^t r(\tau)\, d\tau$, so $m(t)$ is the mean number of demands in in the interval $[0,t]$. Then for $k = 0, 1, 2, \ldots$,

$$\text{prob}\{k \text{ demands in } [0,t]\} = e^{-m(t)}\frac{[m(t)]^k}{k!} \quad ;$$

likewise, for $\tau < t$,

$$\text{prob}\{0 \text{ demands in } [\tau,t]\} = e^{-\int_\tau^t r(\sigma)\, d\sigma}$$

Since these facts are consequences of the Poisson assumption and follow by generalizing standard arguments‡ to the case $r = r(t)$, not constant, they will not be proven here.

† Terms which go to zero faster than $\Delta t \to 0$.

‡ see, for example, Haight [2].

Combining the above with ( i) and taking the limit as $\Delta t$ approaches zero, we find

$$\text{prob } \{\text{first demand at } \tau\} = e^{-\int_\tau^t r(\sigma)\, d\sigma}\, r(\tau)\, d\tau .$$

The allocation strategy "one unit to each demand" has a positive probability that all demands encountered will be met successfully. Let $P(n, t)$ be the probability that all demands encountered in $[0,t]$ are filled successfully when one unit is allotted to each demand and $n$ units are available at time $t$. Then

$$P(o, t) \equiv \text{prob } \{0 \text{ demands in } [0, t]\} = e^{-m(t)},$$

$$\begin{aligned} P(1, t) &\equiv \text{prob } \{0 \text{ demands in } [0, t]\} \\ &\quad + p \cdot \text{prob } \{1 \text{ demand in } [0, t]\} \\ &= e^{-m(t)} + p \cdot e^{-m(t)}\, m(t) = e^{-m(t)}\, [1 + pm(t)], \end{aligned}$$

and we can state the following.

Theorem 2.1.1. If $n$ units, each with probability $p$ of functioning successfully, are (a) available when $t$ units of time remain, and (b) allotted one to each demand encountered until zero units of time remain, then the probability that all demands encountered are filled, $P(n, t)$, is given by

$$P(n, t) = e^{-m(t)} \sum_{i=0}^{n} \frac{[pm(t)]^i}{i!}$$

Proof: The formula holds for $n = 0, 1$. Assume it holds for $n \leq k$. Then

$$\begin{aligned} P(k+1, t) &= e^{-m(t)} + \int_0^t p^{P(k,\tau)} e^{-\int_\tau^t r(\sigma)\, d\sigma}\, r(\tau)\, d\tau \\ &= e^{-m(t)} + \int_0^t \{\sum_{i=0}^{k} \frac{[pm(\tau)]^i}{i!}\}\, e^{-m(\tau)}\, e^{-\int_\tau^t r(\sigma)\, d\sigma}\, pr(\tau)\, d\tau \\ &= e^{-m(t)} + e^{-m(t)} \int_0^t \{\sum_{i=0}^{k} \frac{[pm(\tau)]^i}{i!}\}\, pr(\tau)\, d\tau \\ &= e^{-m(t)} \{1 + \int_0^{pm(t)} \sum_{i=0}^{k} \frac{u^i}{i!}\, du\} \\ &= e^{-m(t)} \{1 + \sum_{i=0}^{k} \frac{[pm(t)]^{i+1}}{(i+1)!}\} \\ &= e^{-m(t)} \sum_{i=0}^{k} \frac{[pm(t)]^i}{i!} \end{aligned}$$

A larger probability that all demands encountered are filled successfully can be obtained by allocating more than one unit under some circumstances. Let $P^*(n, t)$ be the maximum probability of successfully meeting all demands in $[0, t]$ given $n$ units available when demands may occur for $t$ more units of time. That is, $P^*(n, t)$ is the probability of meeting all demands in $[0, t]$ when the allocatio strategy used maximizes this probability. Since the strategies are identical when $n = 0$ and $n = 1$ we have

$$P^*(0, t) = P(0, t)$$

$$P^*(1, t) = P(1, t) .$$

In general,

$$P^*(n, t) = e^{-m(t)} + \int_0^t \max_{i=1(1)n} \{(1 - q^i) P^*(n-i, \tau)\} e^{-\int_\tau^t r(\sigma)\, d\sigma} r(\tau)\, d$$

The integration is over all possible first demand times, $\tau$; $q$ denotes $1-p$, the probability that an allotted unit fails to fill the demand. If units are allotted, the probability of filling the first demand is $(1-q^i)$. The "optimality principle" and independence yield the above. That $P^*(n, t)$ depends only on $m(t)$ and not explicitly on $t$ is the point of the following theorem.

Theorem 2. 1. 2. $P^*(n, t) = e^{-m(t)} \Phi(n, m(t))$,

where $\Phi(n, x)$ is defined by

$$\Phi(n, x) \equiv 1 + \int_0^x \max_{i=1(1)n} \{(1-q^i) \Phi(n - i, y)\}\, dy \quad \text{for } n \geq 1$$

and

$$\Phi(0, x) \equiv 1 .$$

Proof: The theorem is true for $n = 0$. Assume it is true for $n < k$. Then

$$P^*(k, t) = e^{-m(t)} + \int_0^t \max_{i=1(1)k} \{(1 - q^i) P^*(k-i, \tau)\} e^{-\int_\tau^t r(\sigma)\, d\sigma} r(\tau)\, d\tau$$

$$= e^{-m(t)} + \int_0^t \max_{i=1(1)k} \{(1 - q^i) e^{-m(\tau)} \Phi(k-i, m(\tau))\} e^{-\int_\tau^t r(\sigma)\, d\sigma} r(\tau)$$

$$= e^{-m(t)} [1 + \int_0^{m(t)} \max_{i=1(1)k} \{(1-q^i) \Phi(k-i, y)\}\, dy]$$

$$= e^{-m(t)} \Phi(k, m(t)) .$$

Thus $P^*(n, t)$ can be tabulated versus $n$ and $m(t) = x$. To simplify notation, henceforth we write $P(n, x)$ instead of $P^*(n, t)$ for the probability that all demands are successfully filled given that $n$ unit of supply are available when $x = m(t)$ demands are expected.

By the preceding theorem the optimum† number of units to allo cate to a demand is given by that $i$ which attains the maximum in the

†Relative to the criterion "maximize probability of successfully meetir all demands".

integrand defining $\Phi(n,x)$. To render this unique, we define the allocation strategy function $\Psi(n,x)$ as the smallest value of $i$ at which $(1-q^i)\cdot\Phi(n-i,x)$, considered as a function of $i$, achieves its maximum. Using prime to denote differentiation with respect to $x$, symbolically,

$$\Psi(n,x) = \text{first}\,[\,i = 1(1)n:\ (1-q^i)\cdot\Phi(n-i,x) = \Phi'(n,x)\,]$$

and

$$\Phi(n,x) = 1 + \int_0^x [1-q^{\Psi(n,y)}]\cdot\Phi(n-\Psi(n,y),y)\,dy,$$

where "first $[\,i = 1(1)n: \ldots]$" denotes the first value of $i$ from $1$ to $n$, in steps of $1$, at which the condition to the right of the colon holds.

For $n = 1$ and $2$, direct application of the above yields:

$$\Phi(1,x) = 1 + (1-q)x = 1 + px,$$

$$\Psi(1,x) = 1,$$

and

$$\Phi(2,x) = \begin{cases} 1 + (1-q^2)x, & x \le \dfrac{q}{1-q} = \dfrac{q}{p} \\ 1 + \dfrac{q^2}{2} + px + \dfrac{p^2x^2}{2}, & x > \dfrac{q}{p}\,^{\dagger} \end{cases}$$

$$\Psi(2,x) = \begin{cases} 2, & x < q/p, \\ 1, & x \ge q/p \end{cases}.$$

Thus for $n$ small the $\Phi(n,x)$ are polynomials of degree $\le n$ on certain intervals of the $x$-axis. The $\Phi$ and $\Psi$ functions for $n = 3$ and $4$ can also be easily given in analytic form:

$$\Phi(3,x) = \begin{cases} 1 + (1-q^3)x, & 0 \le x < \dfrac{q^2}{1-q^2}, \\ 1 + \dfrac{q^4}{2(1+q)} + (1-q^2)x + \dfrac{(1-q^2)(1-q)x^2}{2}, & \dfrac{q^2}{1-q^2} \le x < \dfrac{q+\sqrt{2q}}{1-q}, \\ 1 + \dfrac{q^4}{2(1+q)} + q^2 + \dfrac{2q\sqrt{2q}}{3} - \dfrac{q^3}{3!} & \\ \quad + (1 + \dfrac{q^2}{2})(1-q)x + \dfrac{(1-q)^2x^2}{2} + \dfrac{(1-q)^3x^3}{3!}, & \dfrac{q+\sqrt{2q}}{1-q} \le x; \end{cases}$$

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$\dagger$ in this case the integral has to be broken up in two parts as follows:

$$\phi(2,x) = 1 + \int_0^{q/p} (1-q^2)\,dy + \int_{q/p}^x p(1+py)\,dy$$

$$= 1 + (1+q)q + p(x - \frac{q}{p}) + p^2(\frac{x^2}{2} - \frac{(q/p)^2}{2}).$$

$$\Psi(3,x) = \begin{cases} 3, & 0 \le x < \dfrac{q^2}{1-q^2} \\[2ex] 2, & \dfrac{q^2}{1-q^2} \le x < \dfrac{q+\sqrt{2q}}{1-q} \\[2ex] 1, & \dfrac{q+\sqrt{2q}}{1-q} \le x\,; \end{cases}$$

$$\Phi(4,x) = \begin{cases} 1 + (1-q^4)x, & 0 \le x < \dfrac{q^3}{1-q^3} \\[2ex] 2 - \dfrac{q^4(1+q+q^2/2)}{1+q+q^2} + (1-q^3)x + \dfrac{(1-q^3)(1-q)x^2}{2}, & \dfrac{q^3}{1-q^3} \le x < \dfrac{q}{1-q} \\[2ex] 2 - \dfrac{q^4(1+q+q^2/2)}{1+q+q^2} + \dfrac{q^3}{3}(1-\dfrac{q}{2}) + (1-q^2)(1+\dfrac{q^2}{2})x + \dfrac{(1-q^2)(1-q}{2} & \\[2ex] \qquad + \dfrac{(1-q^2)(1-q)^2x^3}{3!}, & \dfrac{q}{1-q} \le x < c( \\[2ex] 2 - \dfrac{q^4(1+q+q^2/2)}{1+q+q^2} + \dfrac{q^3}{3}(1-\dfrac{q}{2}) + q(1-\dfrac{q}{2}-\dfrac{q^3}{2})c(q) & \\[2ex] \qquad + [\dfrac{1}{2}+q-\dfrac{q^2}{4}](1-q)^2c^2(q) + \dfrac{q(1-q)^3c^3(}{3!} & \\[2ex] \qquad - \dfrac{(1-q)^4c^4(q)}{4!} + (1-q)x + (1+\dfrac{q^2}{2})(1-q)^2\dfrac{x}{2} & \\[2ex] \qquad + \dfrac{(1-q)^3x^3}{3!} + \dfrac{(1-q)^4x^4}{4!}, & c(q) \le x\,; \end{cases}$$

$$\Psi(4,x) = \begin{cases} 4, & 0 \le x < \dfrac{q^3}{1-q^3}\,, \\[2ex] 3, & \dfrac{q^3}{1-q^3} \le x < \dfrac{q}{1-q}\,, \\[2ex] 2, & \dfrac{q}{1-q} \le x < c(q)\,, \\[2ex] 1, & c(q) \le x\,; \end{cases}$$

where

$$c(q) = \frac{1}{1-q}\left[ q + (h(q) + f(q))^{1/3} + (h(q) - f(q))^{1/3}\right]$$

$$h(q) = q\left( \frac{3}{2} q \frac{(1+2q)}{1+q} + 3 - 2\sqrt{2q} \right),$$

$$f(q) = \sqrt{(h(q))^2 - 8q^3}\ .$$

(The formula for the roots of a cubic equation was used in obtaining $\Phi(4, x)$, $c(q)$, $h(q)$, and $f(q)$. See [3], p. 358.)

As one can see, for higher values of $n$ the complexity of $\Phi(n, x)$ increases markedly. Nevertheless, extensive tables of $P(n, x) = e^{-x}\Phi(n, x)$ have been computed at The RAND Corporation by use of approximate numerical integration in digital computer programs.

As a practical matter, it is quite easy to compute the functions $\Phi$ numerically. Such computation simply involves the solution of a system of, say, $N + 1$ differential equations of the form

$$\Phi'(n, x) = \max_{i = 1(1)n} \{(1-q^i)\Phi(n-i, x)\} \quad n = 0, 1, \ldots, N,$$

subject to the initial conditions

$$\Phi(n, 0) = 1 \quad n = 0, 1, \ldots, N\ .$$

Many of the properties of the functions $\Phi(n, x)$ and $\Psi(n, x)$ which hold for the analytic expressions also hold, in general, for all $n$. The following section presents a rigorous mathematical derivation of these properties, based solely on the above definitions. Many of these properties correspond to what our intuition tells us that holds, given the physical interpretation of the model. However, some results which are intuitively almost obvious, and which are verified by the tables referred to above, remain as yet unproven.

2.2. Properties of the Functions $\Phi$ and $\Psi$. First, let us make the dependence of $\Phi(n, x)$ on the parameter $0 \le q \le 1$ explicit by writing $\Phi(q, n, x)$.

Lemma 2.2.1. If $n \ge 1$, $x > 0$, and $0 \le q < q' \le 1$, then

$$\Phi(q, n, x) > \Phi(q', n, x)\ .$$

Proof: Assume $n = 1$. Then

$$\Phi(q, 1, x) = 1 + (1-q)x > 1 + (1-q')x = \Phi(q', 1, x)\ .$$

Assume the lemma holds for all $n \ge 1$ and $n < k$, then since

$$(1-q^n)\Phi(q, k-n, x) > (1-q'^n)\Phi(q', k-n, x)$$

for all $1 < n \le k$, the desired result follows at once from the definition of $\Phi$.

Lemma 2.2.2. $\Phi(1, n, x) = 1$.

Proof: Trivial.

The case $q = 1$ (that is, all units have zero success probabili is exceptional, and in subsequent discussions we shall assume $q < 1$ unless we explicitly state otherwise.

Corollary 2.2.1. $\Phi(n, x) \geq 1$.

Lemma 2.2.3. $\Phi(0, n, x) = \sum_{i=0}^{n} \frac{x^i}{i!}$.

Proof: The statement is true for $n = 0$, and the inductive ste trivial.

Corollary 2.2.2. $\Phi(n, x) \leq \sum_{i=0}^{n} \frac{x^i}{i!}$.

Lemma 2.2.4. $\Phi(n, x)$ has a continuous first derivative and $\Phi'(n,x) > 0$ if $n > 1$.

Proof: From the definition we compute

$$\Phi'(n, x) = \max_{i=1(1)n} (1-q^i)\, \Phi(n-i, x).$$

The quantity on the right is clearly positive by Corollary 2.2.

Lemma 2.2.5. $\Phi'(n, x) > \Phi'(n-1, x)$ if $n \geq 1$ and either x : or $q > 0$.

Proof: For $q = 0$, Lemma 2.2.3 above immediately implies the desired result. If $q > 0$, then since

$$(1-q^i)\,\Phi(n-1-i, x) < (1 - q^{i+1})\,\Phi(n-1-i, x) ,$$

it follows that

$$\begin{aligned} \Phi'(n-1, x) &= \max_{i=1(1)n-1} (1-q^i)\,\Phi(n-1-i, x) \\ &< \max_{i=1(1)n} (1-q^i)\,\Phi(n-i, x) = \Phi'(n, x). \end{aligned}$$

Corollary 2.2.3. $\Phi(n, x) > \Phi(n-1, x)$ if $x > 0$.

Lemma 2.2.6. $(1-q^n)\, \Phi(n, x) \geq \Phi'(n, x)$.

Proof: For $q = 0$ this is a trivial consequence of Lemma 2. For $q > 0$ we know that

$$(1-q^n)\,\Phi(n, x) \geq (1-q^i)\,\Phi(n-i, x) \quad \text{for } i = 1\,(1)\,n,$$

and thus the desired result follows.

Corollary 2.2.4. $\Phi(n, x) \leq e^{(1-q^n)x}$.

Proof: By Lemma 2.2.6. we have

$$\frac{\Phi'(n, x)}{\Phi(n, x)} \leq 1-q^n .$$

Integration from $0$ to $x$ gives

$$\ln \Phi(n, x) \leq (1-q^n)\, x .$$

Hence,

$$\Phi(n, x) \leq e^{(1-q^n)x} .$$

From this point on we shall tacitly assume that $0 < q < 1$, anc supress dependence of $\Phi$ and $\Psi$ upon $x$ when it is convenient.

Lemma 2. 2. 7. $\Psi(n,x) \le \Psi(n-1,x) + 1$ if $n \ge 1$.

Proof: Since $\Psi(n-1,x)$ is defined to be the first $i$ for which the function $(1-q^i)\Phi(n-1-i,x)$ attains its maximum we have

$$(1-q^{\Psi(n-1)})\Phi(n-1-\Psi(n-1)) \ge (1-q^{\Psi(n)-1})\Phi(n-\Psi(n)).$$

If $\Psi(n,x) > \Psi(n-1,x) + 1$, then by definition of $\Psi(n,x)$ we have

$$(1-q^{\Psi(n)})\Phi(n-\Psi(n)) > (1-q^{\Psi(n-1)+1})\Phi(n-\Psi(n-1)-1).$$

Combining these inequalities with the identity

$$(1-q^{\Psi(n-1)+1})\Phi(n-\Psi(n-1)-1) = \frac{(1-q^{\Psi(n-1)+1})}{(1-q^{\Psi(n-1)})}(1-q^{\Psi(n-1)})\Phi(n-1-\Psi(n-1))$$

given us

$$(1-q^{\Psi(n)})\Phi(n-\Psi(n)) > \frac{(1-q^{\Psi(n-1)+1})}{(1-q^{\Psi(n-1)})}(1-q^{\Psi(n)-1})\Phi(n-\Psi(n))$$

which implies

$$\frac{1-q^{\Psi(n)}}{1-q^{\Psi(n)-1}} > \frac{1-q^{\Psi(n-1)+1}}{1-q^{\Psi(n-1)}} .$$

Since $(1-q^{y+1})/(1-q^y)$ is a monotone decreasing function of $y$ (for $0<q<1$), this last inequality contradicts the assumption that $\Psi(n,x) > \Psi(n-1,x) + 1$.

In order to derive our main theorem concerning $\Phi$ and $\Psi$, we will need the following general result.

Lemma 2. 2. 8. If $f$, $g$, and $h$ are positive functions defined on $[0,\infty)$, $f = gh$, $h(0) > 1$, and $h$ is a nondecreasing function, then

$$\frac{1 + \int_0^x f(t)\, dt}{1 + \int_0^x g(t)\, dt}$$

is an increasing function of $x$.

Proof: For compactness of notation, we suppress the dummy variable $t$:

$$\frac{1+\int_0^{x+\Delta} f}{1+\int_0^{x+\Delta} g} - \frac{1+\int_0^x f}{1+\int_0^x g} = \frac{(1+\int_0^{x+\Delta} f)(1+\int_0^x g) - (1+\int_0^x f)(1+\int_0^{x+\Delta} g)}{(1+\int_0^{x+\Delta} g)(1+\int_0^x g)}.$$

The denominator is obviously positive, so let us consider the numerator:

$$(1+\int_0^{x+\Delta} f)(1+\int_0^x g) - (1+\int_0^x f)(1+\int_0^{x+\Delta} g)$$

$$= \int_x^{x+\Delta} f\,[1+\int_0^x g] - \int_x^{x+\Delta} g\,[1+\int_0^x f]$$

$$\geq h(x) \int_x^{x+\Delta} g[1+\int_0^x g] - \int_x^{x+\Delta} g[1+\int_0^x f] > 0 .$$

It seems intuitively obvious that $\Psi(n,x) \geq \Psi(n-1,x)$; that is, with a larger supply one is always willing to make at least as generou an allocation. The extensive tables we computed have confirmed this conjecture. However, determined efforts by a number of people at RAND have failed to yield a rigorous proof that this is indeed the case If this could be proven, we could relax the hypothesis in our main the rem below.

Theorem 2.2.1. Suppose $\Psi(k,x) \geq \Psi(k-1,x)$ for all $x$ and for all $k \leq n$. Then the following hold for $n$:

(a) $\Psi(n,x)$ is a monotone nonincreasing function of $x$;

(b) $\Psi(n,x)$ has $n-1$ discontinuities;

(c) $\Phi(n,x)/\Phi(n-1,x)$ is a monotone increasing function of

Proof: Direct computation shows that both (a) and (b) holc for $n = 1$ and $n = 2$. Assume condition (a) holds for all $n < N$. For notational economy, let $i = \Psi(N,x)$. Thus, by the definition of $\Psi(n$ we have

$$(1-q^{i+1})\Phi(N-i-1,x) \leq (1-q^i)\Phi(N-i,x) .$$

Let $j = \Psi(N-i-1,x)$ and $k = \Psi(N-i,x)$. By hypothesis we kn that $i \geq j$:

$$\begin{aligned}
(1-q^{i+1})\Phi'(N-i-1,x) &= (1-q^{i+1})(1-q^j)\Phi(N-i-j-1,x) && \text{(def. of j)}\\
&\leq (1-q^i)(1-q^{j+1})(\Phi(N-i-j-1,x) && \text{(since } i \geq j)\\
&\leq (1-q^i)(1-q^k)\Phi(N-i-k,x) && \text{(def. of k)}\\
&= (1-q^i)\Phi'(N-i,x)
\end{aligned}$$

where prime denotes, as before, differentiation with respect to $x$. This means that if $(1-q^i)\Phi(N-i,x)$ is greater than or equal to $(1-q^{i+1})\Phi(N-(i+1),x)$, then it is also greater than or equal to it for a $y > x$. In other words, $\Psi(N,x)$ can never jump up from $i$ to $i+1$ a: $x$ increases. On the other hand, by the hypothesis of induction $\Psi(N-1,x)$ is monotone decreasing in $x$ and by Lemma 2.2.7 it follo that it is impossible for $\Psi(N,x)$ to jump from $i$ up to any number greater than $i+1$. This completes the proof that $\Psi(N,x)$ is a monc tone nonincreasing function of $x$.

Part (b) of the theorem follows by induction. Direct computa tion shows that (b) holds for $n = 1, 2$. Assume it holds for all $n < N$.

By the expression on p. 5,

$$\frac{\Phi(1,x)}{x} \to 1-q \quad \text{as} \quad x \to \infty$$

Assume for $n < N$

$$\frac{\Phi(n,x)}{x^n} \to \frac{(1-q)^n}{n!} \quad \text{as} \quad x \to \infty .$$

Then

$$\lim_{x\to\infty} \frac{\Phi(N,x)}{x^N} = \lim_{x\to\infty} \frac{1 + \int_0^x \max_{i=1(1)N} \{(1-q^i)\Phi(N-i,y)\}dy}{x^N}$$

$$= \lim_{x\to\infty} \frac{\max_{i=1(1)N} (1-q^i)\Phi(N-i,x)}{Nx^{N-1}}$$

$$= \frac{(1-q)}{N} \frac{(1-q)^{N-1}}{(N-1)!}$$

(The last equality holds since $\Phi(n,x)$ is at most of degree $n$ in $x$.) Hence the inductive step holds and we have $\Phi(n,x)/x^n \to (1-q)^n/n!$ as $x \to \infty$. This implies that for $x$ sufficiently large, $\Psi(N,x) = 1$. It is also easy to see that for $x$ sufficiently small, $\Psi(N,x) = N$. Thus $\Psi(N,x)$ can have at most $N-1$ discontinuities. The only way in which $\Psi(N,x)$ could have fewer than $N-1$ discontinuities is if it drops 2 or more at one of them. By Lemma 2.2.7 and the hypothesis of induction, this could not happen at a point of continuity of $\Psi(N-1,x)$. Let $y$ be a point of discontinuity of $\Psi(N-1,x)$, and say $\Psi(N-1,y) = k$. Then

$$(1-q^k)\Phi(N-k-1,y) = (1-q^{k+1})\Phi(N-k-2,y) .$$

The only way $\Psi(N)$ could drop two at $y$ is if $\Psi(N,y) = k$ and $\Psi(N, y-\epsilon) = k+2$ for $\epsilon$ sufficiently small. This would imply

$$(1-q^k)\Phi(N-k,y) = (1-q^{k+2})\Phi(N-k-2,y) .$$

Eliminating $\Phi(N-k-2,y)$ between these two equations gives

$$\frac{1-q^{k+2}}{1-q^{k+1}} \Phi(N-k-1,y) = \Phi(N-k,y) .$$

Since $(1-q^{k+1})/(1-q^k)$ is a monotone decreasing function of $k$ we have

$$\frac{1-q^{k+2}}{1-q^{k+1}} < \frac{1-q^{k+1}}{1-q^k} .$$

It follows that

$$(1-q^{k+1})\ \Phi(N-(k+1), y) > (1-q^{k})\ \Phi(N-k, y),$$

which implies $\Psi(N, y) = k+1$. This concludes the proof of part (b).

Now we turn to part (c) of our theorem. Part (c) is clearly true for $n = 1, 2$; assume it is true for all $n < N$. Let $i = \Psi(N-1, x)$. We have $\Phi'(n,x) > 0$ if $n > 1$ (Lemma 2.2.4). Then

$$\begin{aligned}\frac{\Phi'(N,x)}{\Phi'(N-1,x)} &= \frac{\max\{(1-q^i)\Phi(N-i,x),\ (1-q^{i+1})\Phi(N-i-1,x)\}}{(1-q^i)\Phi(N-i-1,x)} \\ &= \max\left\{\frac{\Phi(N-i,x)}{\Phi(N-i-1,x)},\ \frac{1-q^{i+1}}{1-q^i}\right\} \\ &= H(N,x) \quad \text{(this is our definition of } H\text{).}\end{aligned}$$

It is clear that $H(N,x) > 1$. By Lemma 2.2.8, all we must do is to show that $H(N,x)$ is a nondecreasing function of $x$ and our proof will be complete. It is clear from the hypothesis of induction that $H(N,x)$ is nondecreasing at points of continuity of $i=\Psi(N-1,x)$. Let $y$ be a point of discontinuity of $\Psi(N-1,x)$, and let $\Psi(N-1,y) =$ then for all sufficiently small $\epsilon$ we have $\Psi(N-1, y-\epsilon) = k+1$. By definition of $\Psi(N-1, y-\epsilon)$ we then have

$$(1-q^{k+1})\Phi(N-k-2,\ y-\epsilon) > (1-q^k)\Phi(N-1-k,\ y-\epsilon)$$

$$\therefore\ \frac{\Phi(N-(k+1),\ y-\epsilon)}{\Phi(N-(k+1)-1,\ y-\epsilon)} < \frac{1-q^{k+1}}{1-q^{k}}\ .$$

Since

$$\frac{1-q^{k+2}}{1-q^{k+1}} < \frac{1-q^{k+1}}{1-q^{k}}\ ,$$

it follows that $H(N,y) > H(N, y-\epsilon)$ for all sufficiently small $\epsilon$, wh completes the proof.

<u>2.3. Discussion</u> The first few results of Sec. 2.2 deal with the pa meter $q = 1-p$, the probability that a unit fails to function. Lemma 2. states that for a given number of units $n$, less reliable units yield a lower maximum probability of meeting all demands; hence from the ex treme cases $q = 0$ and $q = 1$, bounds can be obtained for arbitrary Lemma 2.2.2 states that totally unreliable units do not increase the probability of meeting all demands above that of the event "no demand occur". Lemma 2.2.3 shows that with $n$ perfectly reliable units ($p = 1$) $\Phi$ is the sum of the first $n+1$ terms of the power series for $e^x$. From the preceding we get the bounds of Corollaries 2.2.1 and 2.2.2, which imply

$$e^{-x} < P(n,x) < e^{-x}\sum_{i=0}^{n}\frac{x^i}{i!}\ .$$

An alternate upper bound on $P(n, x)$ follows from results on $\Phi'(n, x)$, the derivative with respect to $x$. From Corollary 2.2.4 we obtain the bound

$$P(n, x) \leq e^{-q^n x} ,$$

which is generally smaller than the preceding one for $x$ small; the summation bound is smaller, for $x$ large. The results are:

(1) $\Phi$ has a continuous positive $x$-derivative, which is a monotonic increasing function of $n$ (Lemmas 2.2.4 and 2.2.5).

(2) $\Phi$ is a monotonic increasing function of $n$ so more units being available improves the maximum probability of meeting all demands, $P(n, x)$ (Corollary 2.2.3).

The remaining results concerning the optimum allocation strategy $\Psi(n, x)$ are set forth in Theorem 2.2.1. Part a) confirms that fewer units should be allocated to a single demand when more demands are expected: $\Psi(n, x)$ is a monotonic nonincreasing step function of $x$. When $x$ is sufficiently small (few demands expected) all $n$ units should be allocated to any demand that occurs. The values of $x$ at which $\Psi$ changes can be expressed in terms of $n$ and $q$ as discussed below.

There are two conjectures about the $\Phi$ and $\Psi$ functions which appear plausible and which are supported by extensive computations, but which we have not been able to establish by a rigorous mathematical argument:

(a) $\Psi(n, x) \geq \Psi(n-1, x)$ for all $x \geq 0$ and for all $n \geq 1$.

(b) $\Phi(n, x)/\Phi(n-1, x)$ is a monotone decreasing function of $n$.

Conjecture (b) immediately implies conjecture (a). Note that it is not always the case that the increment $\Phi(n, x) - \Phi(n-1, x)$ is a monotone decreasing function of $n$. For example, when $q = 0.5$, the analytic expressions for $\Phi(n, x)$, $n = 0, 1$, and $2$ (see p. 5) yield

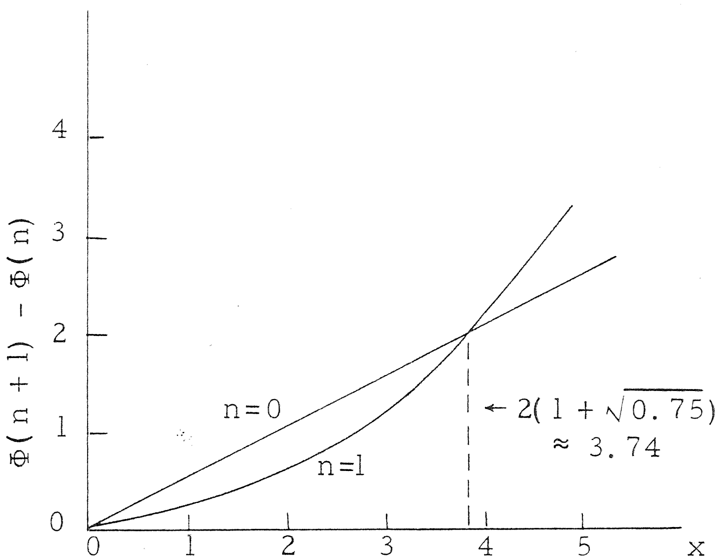
$$\Phi(1) - \Phi(0) = 0.5x ,$$

$$\Phi(2) - \Phi(1) = \begin{cases} 0.25x , & x \leq 1 , \\ 0.125(1+x^2) , & x > 1 . \end{cases}$$

Figure 1 shows the region where the increment in success probability is greater for the second unit than the first.

However, for many values of $n$ and $x$ the increment in $\Phi$ is monotone decreasing. It is easy to show that when these increments are monotone decreasing, then (b) (and hence (a)) holds.

Because of the frequent validity of (a) and its key role in the mathematical development we define it henceforth as the regularity condition for $\Psi$. When this condition holds, it can be proven that the discontinuities of $\Psi(n, x)$ are related to those of $\Psi(n-2, x)$. If $x = y_{n\ell}$ is the $\ell$-th discontinuity of $\Psi(n, x)$, as $x$ increases from




Increments in success probability versus expected number of demands for first and second units.
Figure 1

zero, and $\Psi$ satisfies the regularity condition, then

$$y_{n1} = \frac{q^{n-1}}{1-q^{n-1}},$$

$$y_{n\ell} = y_{n-2,\ell-1}, \quad \text{for } \ell = 2, 3, \ldots, n > 3\ell - 2 .$$

and

$$y_{n\ell} = \frac{q^{n+1-2\ell}}{1-q^{n+1-2\ell}}$$

for $\ell = 2, 3, \ldots, n > 3\ell - 2$

( Proofs of the first two statements are given in Corollary 4. 1. 5 and 4. 1. 6; the third statement is proven in [1], p. 77. )

3. Optimal Policies The purpose of this section is to illustrate the application of the optimum strategy functions $\Psi(n, x)$. We also wish to make explicit that although the optimum return functions $\Phi(n, x)$ ai computed a priori, application of $\Psi$ does in practice yield the best return.

For any sample function of the random process ( demands, succ of allotted units), the optimal policy consists of using different strate functions as the process unfolds. Suppose $n = N$ units are available the initial time $t = T$, and the random variables "total number of de- mands" $L$ and "actual-demand-times" are $T_i$, $i = 1, 2, \ldots, L$, where $T_i < T_{i+1} \leq T_L < T$. Then the allocation at the first demand encountere $T_L$, should be

$$\Psi(N, m(T_L)) = N - N_L, \qquad 0 \leq N_L \leq N-1,$$

where $N_L$ inventory is retained for future demands. If this demand is met, at the second demand

$$\Phi(N_L, m(T_{L-1})) = N_L - N_{L-1}, \quad 0 \leq N_{L-1} \leq N_L - 1,$$

units should be allotted, $N_{L-1}$ units retained, and so forth. (Here the $N_i$, $i \leq L$, are a sequence of random variables since they depend on the actual $T_i$ realized.)

Dreyfus [4] has observed that the best possible stochastic allocation policy is obtained by taking account of the random realization in the optimization, as it occurs. We have done this. We apply the best allocation to the first demand which is to occur, taking account of the time of occurrence as well as of the inventory available. Then, the random inventory remaining, together with the time, $T_{L-1}$, at which the next demand occurs, determines the one-stage-optimal allocation $\Psi(N_L, m(T_{L-1}))$ used at a future demand. (Each one-stage optimal allocation assumes optimal use of the remaining inventory.)

Note that all the information we have about subsequent demands at time $T_k$ is summed up in the number $m(T_k)$; if any additional information were conveyed by the values of the $T_i$'s $(i > k)$, or even by the value of $L - k$, then our problem might be much more difficult, both conceptually and computationally.

## 4. Other Models.

The Poisson demand model can be used in connection with criteria other than probability of meeting all demands. This section extends the probability criterion to a more complex model, and illustrates two additional criteria: inventory remaining after meeting all demands, and consecutive demands filled given that all demands should be met if possible. In the first case (4.1) the results presented are as detailed as for the basic model; however, a less comprehensive analysis is provided for the alternate criteria (4.2, 4.3). These results illustrate how our formulation can be applied to other contexts.

### 4.1. Unreliable Demands.

In this section we assume that a demand may be withdrawn with probability $w = 1-v$, and that this is independent of the number of units allotted, the occurrence of demands, and the success of whatever units have been allotted. We let $P(n, t, v)$ be the maximum probability of filling all demands encountered with time $t$ remaining and $n$ units available† (in other words the probability of success given that units are used to maximize probability of success).

#### 4.1.1. The Model When Demand May Be Withdrawn.

Since an inventory of no units can "meet all demands encountered" provided that all demands encountered are withdrawn, we have

$$P(0,t,v) = e^{-m(t)} + e^{-m(t)} m(t) \cdot w + e^{-m(t)} \frac{[m(t)]^2}{2!} \cdot w^2 + \dots$$

$$= e^{-m(t)} \cdot \sum_{n=0}^{\infty} \frac{[m(t) \cdot w]^n}{n!} = e^{-m(t)} e^{w \cdot m(t)},$$

$$P(0,t,v) = e^{-v \cdot m(t)}.$$

† This is analogous to the earlier quantity $P^*(n, t)$

If $i$ unreliable units are allotted to an unreliable demand which has been encountered, the probability that these units fill that demand is

$$\text{prob }\{\text{demand withdrawn or if not withdrawn then at least one unit works}\} = w + (1-w)(1-q^i) = 1-vq^i \; .$$

Noting that one might choose to allot no units at all, $P(n,t,v)$ is defined in accordance with the analysis in section 2 as

$$P(n,t,v) \equiv e^{-m(t)} + \int_0^t \max_{i=0(1)n} \{(1-vq^i)\,P(n-i,\tau,v)\}\, e^{-\int_\tau^t r(\sigma)\,d\sigma}\, r(\tau)\, d\tau,$$

where the factors involving $r$ make up the probability that the first demand is encountered at time $\tau$.

Intuitively, nothing is gained at an encounter if no units are allotted. Indeed a rigorous mathematical proof exists that the maximu is never attained at $i = 0$. We state this as Theorem 4.1.1.

Theorem 4.1.1. At least one unit should be allotted to any unreliable demand encountered, if this is possible.

Proof:† Let $y$ be any arbitrary $t$. By substituting $P(1,t,v)$ from Corollary A.1 of the Appendix and $P(0,t,v) = e^{-v\,m(t)}$ we fin after some manipulation that the following inequality holds for $n = 1$:

$$\text{(i)} \qquad P(n,y,v) < \frac{(1-qv)}{(1-v)}\, P(n-1,y,v)\,.$$

Assume that (i) holds for $n < k$. $P(k,0,v) = P(k-1,0,v) = 1$ hence (i) holds for $y = 0$, and, since both sides are continuous, for an interval about $y = 0$. Assume that there exists a $y_0 > 0$ such that (i) holds for $y < y_0$, with equality at $y_0$. Then for $y < y_0$, we hav by the definition of $P'(k,y,v)$ that

$$P'(k,y,v) = \max_{j=0(1)k} (1-q^j v)\, P(k-j,y,v)$$

$$\text{(ii)} \qquad < \max_{n=0(1)k-1} (1-q^j v)\, \frac{(1-qv)}{(1-v)}\, P(k-1-j,y,v)$$

$$= \frac{1-qv}{1-v}\, P'(k-1,y,v)\;.$$

To prove the inequality (ii), consider it term by term. For $j = 0$, (ii) reduces to (i), and hence is true by assumption for $y < y_0$. For $j = 1, 2, \ldots, k-1$, the inequality (ii) holds by the inductive hypothesis. The term $j = k$ appears only in the left hand side of the inequality sig but

† Due to R. Strauch, The RAND Corporation.

$$(1-q^{k}v)\,P(k-k,y,v) < (1-q^{(k-1)}v)\,\frac{(1-qv)}{(1-v)}\,P(0,y,v)$$

$$= (1-q^{(k-1)}v)\,\frac{(1-qv)}{(1-v)}\,P(k-1-(k-1),y,v)$$

$$\leq \max_{j=0(1)k-1}\left\{(1-q^{j}v)\,\frac{(1-qv)}{(1-v)}\,P(k-1-j,y,v)\right\}.$$

This proves (ii). But since

$$P(k,0,v) < \frac{(1-qv)}{(1-v)}\,P(k-1,0,v)\ ,$$

and, for $y < y_0$,

$$P'(k,y,v) < \frac{(1-qv)}{(1-v)}\,P'(k-1,y,v)\ ,$$

it is impossible that equality holds for $y = y_0$. Hence (i) is true for all $n$, and the theorem follows.

Theorem 4.1.2.

(i) $\quad P(n,t,v) = e^{-m(t)}\,\Phi(n,m(t),v)\ ,$

where $\Phi(n,x,v)$ is defined for $n = 0$

(ii) $\quad \Phi(0,x,v) = e^{wx}\ ,$

and, for $n \geq 1$,

(iii) $\quad \Phi(n,x,v) = 1 + \int_0^x \max_{i=1(1)n}\{(1-vq^{i})\Phi(n-i,y,v)\}\,dy\ .$

Proof: By definition of $P(n,t,v)$ and the preceding theorem,

$$P(n,t,v) = e^{-m(t)} + \int_0^t \max_{i=1(1)n}\{(1-vq^{i})\,P(n-i,\tau,v)\} \cdot e^{\int_\tau^t r(\sigma)\,d\sigma}\,r(\tau)\,d\tau\ .$$

Assume that the assertion is true for $n < k$. Then

$$P(k,t,v) = e^{-m(t)} + \int_0^t \max_{i=1(1)k}\{(1-vq^{i}) \cdot e^{-m(\tau)}\,\Phi(k-i,m(\tau),v)\}e^{-\int_\tau^t r(\sigma)\,d\sigma}\,r(\tau)\,d\tau$$

$$= e^{-m(t)}\left[1 + \int_0^{m(t)} \max_{i=1(1)k}\{(1-vq^{i})\Phi(k-i,y,v)\}\,dy\right].$$

Thus the theorem holds then for $n = k$. Since it holds also for $n = 0$, it is true for all $n$.

Theorem 4.1.2 is consistent with the earlier case where $v = 1$ corresponding to demand never withdrawn. For

$$\lim_{v=1} \Phi(0, x, v) = 1 = \Phi(0, x),$$

and hence we have the following lemma:

Lemma 4.1.1. $\lim_{v \to 1} \Phi(n, x, v) = \Phi(n, x)$ .

Proof: Assume that this is true for $n < k$, and let prime denote differentiation with respect to $x$ at constant $v$ and $n$. We then have

$$\Phi'(k, x, v) \leq \max_{i=1(1)k} (1-vq^i)\, \Phi(k-i, x, v)$$

$$\leq \max_{i=1(1)k} \Phi(k-i, x, v) < e^x$$

since $P(n, m^{-1}(x), x) = e^{-x}\, \Phi(n, x, v) \leq 1$ by the definition of $\Phi(n, x$ Furthermore, we have

$$\lim_{v \to 1} \Phi'(k, x, v) = \lim_{v \to 1} \max_{i=1(1)k} \{(1-vq^i)\, \Phi(k-i, x, v)\}$$

$$= \max_{i=1(1)k} \{(1-q^i)\, \Phi(k-i, x)\} = \Phi'(k, x) .$$

Hence

$$\lim_{v \to 1} \Phi(k, x, v) = \lim_{v \to 1} \int_0^x \Phi'(k, y, v)\, dy + 1$$

$$= \int_0^x \Phi'(k, y)\, dy + 1 = \Phi(k, x)$$

by Lebesgue's theorem on dominated convergence (see, for example, Ref. [5]). The lemma follows, since this proves the induction hypothesis, and we already have seen that the lemma is true for $n = 0$ .

4.1.2. Properties of the Functions $\Phi$ and $\Psi$ for $0 < v < 1$. Let $\Psi(n, x, v)$ be the strategy function for $\Phi(n, x, v)$, defined analogously to $\Psi(n, x)$ of Sec. 2. This part presents the properties whic carry over to the more complex model where demand may be withdrawr following the development of 2.2.

Lemma 4.1.2. If $n \geq 1$, $x > 0$, and $0 \leq q < q' \leq 1$, then

$$\Phi(q, n, x, v) > \Phi(q', n, x, v)$$

for all $0 < v \leq 1$ .

Proof: The assertion is true for $n = 1$. If $v > 0$ and the lemr is assumed true for all $n \geq 1$ and $n < k$, then

$$(1-vq^n)\, \Phi(q, k-n, x, v) > (1-vq'^n)\, \Phi(q', k-n, x, v),$$

all $1 \leq n < k$, and the result follows by definition of $\Phi(\cdot, \cdot, \cdot, v)$.

Lemma 4.1.3. $\Phi(1, n, x, v) = e^{wx}$, all $n$, $0 \leq v \leq 1$ .

Proof: Immediate.

From here on we assume $q < 1$ unless otherwise stated.

Corollary 4.1.1. $\Phi(n, x, v) \geq e^{wx}$ .

Lemma 4. 1. 4.

(i) $\Phi(0,n,x,v) = 1 + \int_0^x \Phi(0,n-1,y,v)\, dy$

(ii) $\Phi'(0,n,x,v) = \Phi(0,n-1,x,v)$,

(ii) $\Phi(0,n-1,x,v) = \max_{i=1(1)n} \Phi(0,n-i,x,v)$ .

Proof: The equivalence of all three statements is immediate by inserting $q = 0$ in the definition of $\Phi(q,n,x,v)$. (iii) holds for $n = 2$ since

$$\Phi(0,0,x,v) = e^{wx} < \frac{e^{wx}-v}{w} = \Phi(0,1,x,v)$$

for $w > 0$, $x > 0$, by Cor. A. 1 of the Appendix. Assume the lemma is true for $n < k$. Then it is true for all $n$, since

$$\Phi(0,k-1,x,v) > \Phi(0,k-2,x,v)$$

because

$$\Phi(0,k-2,0,v) = 1 = \Phi(0,k-1,0,v)$$

and by (ii) and the hypothesis of induction

$$\begin{aligned}\Phi'(0,k-2,x,v) &= \Phi(0,k-3,x,v) < \Phi(0,k-2,x,v)\\ &= \Phi'(0,k-1,x,v)\end{aligned}$$

Lemma 4. 1. 5.

(i) $$\Phi(0,n,x,v) = \frac{e^{wx}}{w^n} + \sum_{r=0}^{n-1}\left(1 - \frac{1}{w^{n-r}}\right)\frac{x^r}{r!} .$$

(ii) $$\Phi(0,n,x,v) = \frac{1}{w^n}[e^{wx} - h(n,wx)] + h(n,x)$$

where $$h(n,x) = \sum_{i=1}^{n}\frac{x^i}{i!} = \Phi(0,n,x) = \lim_{v\to 1}\Phi(0,n,x,v) ,$$

Proof: Let $f(n,x) = \Phi(0,n,x,v)$. Using prime to denote the derivative with respect to $x$, by (ii) of Lemma 4. 1. 4 $f(n,x)$ satisfies

$f'(n,x) = f(n-1,x)$ subject to the conditions

$f(n,0) = 1$,

$f(0,x) = e^{wx}$,

$f(1,x) = \dfrac{e^{wx} - v}{w}$,

and the lemma is the solution of this equation. Since $e^{wx} - h(n,wx)$ is the tail of the exponential power series from $(wx)^{n+1}/(n+1)!$ on, the limit of $\Phi(0,n,x,v)$ is indeed $h(n,x)$ .

Corollary 4.1.2.

$$\Phi(n,x,v) \le \frac{e^{wx}}{w^n} + \sum_{r=0}^{n-1}\left(1-\frac{1}{w^{n-r}}\right)\frac{x^r}{r!} .$$

Lemma 4.1.6. $\Phi(n,x,v)$ has a continuous, positive, first derivative for $n = 0, 1, 2, \ldots$ .

Proof: Analogous to Lemma 2.2.4 for $n > 0$ ; for $n = 0$, by inspection.

Lemma 4.1.7. $\Phi'(n,x,v) > \Phi'(n-1,x,v)$ if $n \ge 1$ and either $x > 0$ or $q > 0$ .

Proof: For $q = 0$ by Lemma 4.1.5,

$$\begin{aligned}
\Phi'(0,n,x,v) &= \frac{e^{wx}}{w^{n-1}} + \sum_{r=1}^{n-1}\left(1-\frac{1}{w^{n-r}}\right)\frac{x^{r-1}}{(r-1)!} \\
&= \frac{e^{wx}}{w^{n-1}} + \sum_{s=0}^{n-2}\left(1-\frac{1}{w^{n-1-s}}\right)\frac{x^{s}}{s!} \\
&> \frac{e^{wx}}{w^{n-2}} + \sum_{s=0}^{n-3}\left(1-\frac{1}{w^{n-2-s}}\right)\frac{x^{s}}{s!} \\
&= \Phi'(0,n-1,x,v) .
\end{aligned}$$

If $q > 0$ the proof parallels that of Lemma 2.2.5.

Corollary 4.1.3. $\Phi(n,x,v) > \Phi(n-1,x,v)$ if $x > 0$ .

Lemma 4.1.8. $(1-q^n v)\,\Phi(n,x,v) \ge \Phi'(n,x,v)$ .

Proof: For $q = 0$ this follows from Corollary 4.1.3. and (ii) of Lemma 4.1.4. For $q > 0$ ,

$$(1-vq^n)\,\Phi(n,x,v) \ge (1-vq^i)\,\Phi(n-i,x,v) \quad \text{for } i = 1(1)n$$

by Corollary 4.1.3, so the result follows.

Corollary 4.1.4. $\Phi(n,x,v) \le e^{(1-q^n v)x}$ .

Proof: Same as proof of Corollary 2.2.4.

Henceforth $0 < q < 1$ unless otherwise stated.

Lemma 4.1.9. $\Psi(n,x,v) \le \Psi(n-1,x,v) + 1$ if $n \ge 1$ .

Proof: Analogous to that of Lemma 2.2.7.

Theorem 4.1.3. If $\Psi(k,x,v)$ is a monotone nondecreasing function of $k$ for $0 \le k \le n$ , then $\Psi(n,x,v)$ is a monotone non-increasing function of $x$ .

Proof: The theorem holds for $n = 1$. It is proven as in part (a) of Theorem 2.2.1. Let all quantities there depends on v also and note that the lemmas cited are also true in this case ( see Lemma 4.1.

Lemma 4.1.10. For $v > 0$, $1 \le j < i + 1$ ,

$$\frac{1-vq^{i+1}}{1-vq^i} < \frac{1-vq^j}{1-vq^{j-1}} .$$

<u>Proof</u>: $-vq^{i+1} - vq^{j-1} < -vq^{i} - vq^{j}$ is equivalent to

$$-q^{i}(1-q) > -q^{j-1}(1-q)$$

or

$$(q^{j-1} - q^{i})\,p > 0\ .$$

<u>Theorem 4.1.4.</u>

(i)
$$\lim_{x\to\infty} \frac{\Phi(n,x,v)}{e^{wx}} = \left(\frac{1-vq}{1-v}\right)^{n} .$$

For $x$ sufficiently large

(ii)
$$\max_{i=1(1)n} \{(1-vq^{i})\,\Phi(n-i,x,v)\} = (1-vq)\,\Phi(n-1,x,v)\ ,$$

or equivalently,

(iii)
$$\Psi(n,x,v) = 1\ .$$

<u>Proof:</u>† (i) is true for $n = 0$. We prove first that (i) for $n < k$ implies (ii) for $n = k$, then that (i) for $k-1$ and (ii) for $k$ implies (i) for $k$. Assume (i) for $n < k$. (ii) holds for $n = k$ if for $1 < i \leq k$

$$\lim_{x\to\infty} \frac{(1-vq^{i})\,\Phi(k-i,x,v)}{(1-vq)\,\Phi(k-1,x,v)} < 1$$

or if

$$\frac{(1-vq^{i})\left(\dfrac{1-vq}{1-v}\right)^{k-i}}{(1-vq)\left(\dfrac{1-vq}{1-v}\right)^{k-1}} < 1$$

(division by $e^{wx}$ and (i)). But this is the same as

(iv)
$$\frac{1-vq^{i}}{1-vq} < \left(\frac{1-vq}{1-v}\right)^{i-1}$$

which holds for $i = 2$ by a trivial computation. If (iv) holds for $i$ it also holds for $i + 1$, for

$$\frac{1-vq^{i+1}}{1-vq}\left(\frac{1-vq^{i}}{1-vq^{i+1}}\right) < \left(\frac{1-vq}{1-v}\right)^{i-1}$$

implies

$$\frac{1-vq^{i+1}}{1-vq} < \left(\frac{1-vq^{i+1}}{1-vq^{i}}\right)\left(\frac{1-vq}{1-v}\right)^{i-1}$$

Thus we must show

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†Due to J. Folkman of the RAND Corporation.

$$\left(\frac{1 - vq^{i+1}}{1 - vq^{i}}\right) \left(\frac{1 - vq}{1 - v}\right)^{i-1} < \left(\frac{1 - vq}{1 - v}\right)^{i}$$

to prove the induction step. But this is the same as

$$\frac{1 - vq^{i+1}}{1 - vq^{i}} < \frac{1 - vq}{1 - v} ,$$

which holds by Lemma 4.1.10. Hence (iv) holds for all $i$ and hence (ii) holds for $n = k$. To prove (i) for $k$, note that (ii) implies that for $x$ sufficiently large,

$$\Phi'(k, x, v) = (1 - vq)\, \Phi(k-1, x, v) ,$$

and apply L'Hospital' rule. Hence the theorem follows for all $n$ by induction on $k = 0, 1, 2, \ldots$ .

Lemma 4.1.11. $\lim_{x \to 0} \Psi(n, x, v) = n$.

Proof: Since $\lim_{x \to 0} \Phi(n, x, v) = 1$ by (i) of Lemma 4.1.4, there exists a $\delta > 0$ such that for $x < \delta$, $r < n$,

$$\Phi(r, x, v) < 1 + \epsilon ,$$

where

$$\frac{1 - q^{n}v}{1 - q^{n-1}v} - 1 > \epsilon > 0 .$$

Then

$$\begin{aligned}(1-q^{n}v)\,\Phi(n-n, x, v) &= (1 - q^{n}v)\, e^{wx} \\ &\geq (1 - q^{n}v) > (1 - q^{n-1}v)(1+\epsilon) \\ &> (1 - q^{n-i}v)\,\Phi(n-i, x, v)\end{aligned}$$

for $1 \leq i < n$, so

$$\Psi(n, x, v) = n .$$

The following results hold under the condition that $\Psi$ is regular, that is

$$\Psi(n, x, v) \geq \Psi(n-1, x, v)$$

for all $n, x,$ and $v$.

Theorem 4.1.5. $\Psi(n, x, v)$ has $n-1$ discontinuities.

Proof: The assertion holds for $n = 1$. Assume it holds for $n < N$. According to (iii) of Theorem 4.1.4. $\lim_{x \to \infty} \Psi(N, x, v) = 1$. By the preceding lemma, $\lim_{x \to 0} \Psi(N, x, v) = N$. Hence, $\Psi(N, x, v)$ has at most $N-1$ discontinuities. Moreover, $\Psi(N, x, v)$ could have fewer than $N-1$ discontinuities only if it dropped two or more at one discontinuity. By Lemma 4.1.9 and the hypothesis of induction, this could not happen at a point of continuity of $\Psi(N-1, x, v)$. Let $y$ be a point of discontinuity of $\Psi(N-1, x, v)$, and say $\Psi(N-1, y, v) = k$. Then

$$(1 - q^k v)\ \Phi(N-k-1, y, v) = (1 - q^{k+1} v)\ \Phi(N - k - 2, y, v)\ .$$

The only way $\Psi(N, x, v)$ could drop two at $y$ is if

$$\Psi(N, y, v) = k$$

and

$$\Psi(N, y-\epsilon, v) = k + 2$$

for $\epsilon$ sufficiently small. This would imply

$$(1 - q^k v)\ \Phi(N - k, y, v) = (1 - q^{k+2} v)\ \Phi(N - k - 2, y, v).$$

Eliminating $\Phi(N - k - 2, y, v)$ between the last two equations gives

$$\frac{1 - q^{k+2} v}{1 - q^{k+1} v}\ \Phi(N - k - 1, y, v) = \Phi(N - k, y, v)\ .$$

But this and

$$\frac{1 - q^{k+2} v}{1 - q^{k+1} v} < \frac{1 - q^{k+1} v}{1 - q^k v} \qquad (\text{Lemma } 4.1.10)$$

imply

$$(1 - q^{k+1} v)\ \Phi(N - (k + 1), y, v) < (1 - q^k v)\ \Phi(N - k, y, v)\ .$$

Hence

$$\Psi(N, y, v) = k + 1\ .$$

Definition 4.1.1. The $\ell$-th discontinuity of $\Psi(n, x, v)$ as $x$ increases from zero is denoted by $y_{n\ell}$.

Corollary 4.1.5.

(a) $$y_{n1} = \frac{1}{1 - v}\ \ell n \left[\frac{1 - q^{n-1} v}{1 - q^{n-1}}\right],$$

(b) $$\lim_{v \to 1} y_{n1} = \frac{q^{n-1}}{1 - q^{n-1}}\ .$$

Proof: By Lemma 4.1.11, Theorem 4.1.3, and Theorem 4.1.5, $x = y_{n1}$ at the first discontinuity of $\Psi(n, x, v)$. This implies that

$$(1 - q^n v)\ \Phi(0, y_{n1}, v) = (1 - q^{n-1} v)\ \Phi(1, y_{n1}, v)$$

or

$$\frac{1 - q^n v}{1 - q^{n-1} v} = \frac{1 - qv}{1 - v} - \frac{vp}{w}\, e^{-wy_{n1}}\ .$$

Part (a) follows from simplifying this equation. To see (b), apply L'Hospital's rule.

Lemma 4.1.12.

$$\Phi(n, x, v) = \frac{(1 - q^n v)\, e^{wx} - v(1 - q^n)}{1 - v}$$

for

$$x < \frac{1}{1-v} \ln \frac{1-q^{n-1}v}{1-q^{n-1}} = y_{n1} .$$

Proof: For $x < y_{n1}$

$$\Phi(n, x, v) = 1 + \int_0^x (1 - q^n v) e^{wy} dy .$$

Corollary 4. 1. 6.

(a) $$y_{n1} < y_{n-1,1} ,$$

(b) $$y_{n2} = y_{n-2,1} \quad \text{for } n \geq 4 .$$

Proof: (a) follows from Corollary 4. 1. 5, the monotonicity of the exponential, and reduction of this inequality to $q^{n-1} < q^{n-2}$. To see (b) note that for $n \geq 4$ and $x < y_{n-2,1}$, by (a), $\Phi(2, x, v)$ is given by Lemma 4. 1. 12. Then Theorems 4. 1. 3, 4. 1. 5, and Lemma 4. 1. 11 imply that $y_{n2}$ occurs at

$$(1 - q^{n-1}v) \Phi(1, y_{n2}, v) = (1 - q^{n-2}v) \Phi(2, y_{n2}, v) .$$

Hence

$$(1-q^{n-1}v)[(1-qv)e^{wy_{n2}} - vp] = (1-q^{n-2}v)[(1-q^2v)e^{wy_{n2}} - v(1-q^2)] ,$$

which reduces to

$$y_{n2} = \frac{1}{1-v} \ln \frac{1-q^{n-3}v}{1-q^{n-3}} = y_{n-2,1} .$$

Theorem 4. 1. 6. $\Phi(n, x, v)/\Phi(n-1, x, v)$ is a monotone increasing function of $x$.

Proof: For $n = 1$ the assertion holds, since

$$\frac{\Phi(1, x, v)}{\Phi(0, x, v)} = \frac{1 - vq}{1 - v} - \frac{vp}{w} e^{-wx} .$$

Assume it holds for all $n < N$. Let $i = \Psi(N-1, x, v)$. Then define $H(N, x, v)$ by

$$\frac{\Phi'(N, x, v)}{\Phi'(N-1, x, v)} = \frac{\max\{(1-q^i v)\Phi(N-i, x, v),\ (1-q^{i+1}v)\Phi(N-i-1, x, v)\}}{(1-q^i v)\ \Phi(N-i-1, x, v)}$$

$$= \max\left\{\frac{\Phi(N-i, x, v)}{\Phi(N-i-1, x, v)},\ \frac{1-q^{i+1}v}{1-q^i v}\right\} = H(N, x, v) ,$$

$H(N, x, v) > 1$ for all $x$. Hence Lemma 2. 2. 8 can be applied provided $H(N, x, v)$ is a nondecreasing function of $x$. By the induction hypothesis, $H(N, x, v)$ is nondecreasing at points of continuity of $i = \Psi(N-1, x, v)$. Let $y$ be a point of discontinuity of $\Psi(N-1, x, v)$, and let $\Psi(N-1, y, v) = k$. For $\epsilon$ sufficiently small, $\Psi(N-1, y-\epsilon, v) = k + 1$, and $(1-q^{k+1}v)\Phi(N-k-2, y-\epsilon, v) > (1-q^k v)\Phi(N-1-k, y-\epsilon, v)$.

Hence

$$\frac{\Phi(N-(k+1), \ -\epsilon, v)}{\Phi(N-(k+1)-1, y-\epsilon, v)} < \frac{1-q^{k+1}v}{1-q^{k}v} .$$

Since

$$\frac{1-q^{k+2}v}{1-q^{k+1}v} < \frac{1-q^{k+1}v}{1-q^{k}v} ,$$

it follows that $H(N, y, v) > H(N, y-\epsilon, v)$ for all $\epsilon$ sufficiently small, so $H(N, x, v)$ is nondecreasing. Therefore Lemma 2.2.8 proves the theorem.

4.2. Allocation to Preserve Inventory. Seeking to "preserve inventory" corresponds to adding a nonrandom demand at $t = 0$ for as many units as possible. If $t \neq 0$, that demand can be met only if first all random demands encountered on $(0, t)$ are filled. There are two ways to view this. The first divides the total inventory, reserving some units for the terminal demand and operating in accord with the $\Psi$ functions with the remaining units; this approach is well suited to inventories involving two different types of goods. The alternate model, which corresponds to one type of good, seeks the optimum allocation when the total inventory can be used for either random or terminal demands. This model is developed below.

If two distinct goods are involved, the maximum number of units might be limited, for example, by weight or volume constraints. For a total of $L$ units allowed, let $n^* = n(L, x)$ be the optimum amount of inventory set aside to meet random demands. Then $n^*$ can be obtained from the probabilities $P(n, x)$ (or $P(n, x, v)$) by:

$$n^* = n(L, x) = \text{first}\left[ n = 1(1)L : (L-n)P(n,x) = \max_{i=1(1)L} \{(L-i)\ P(i,x)\}\right] .$$

Here $(L-n)P(n, x)$ is the expected number of units delivered to the fixed demand having filled all random demands encountered.

A major purpose of the allocation model is to present material relevant to the overall effectiveness of several alternate inventory configurations. Consequently, we turn now to the case where the inventory consists of only one type of good, a dual-purpose unit useful for both random and terminal demands. The relative merit of this approach compared to the one with two distinct goods is developed in this section. The section is composed of three parts: 4.2.1, the mathematical formulation of the dual-purpose-unit model; 4.2.2, comparison of this type of inventory to the two-distinct-goods configuration analyzed above; and 4.2.3, discussion of a suboptimal strategy for the dual-purpose-unit case.

4.2.1. Mathematical Formulation. Recall that the probability of no demand-encounters in $t$ remaining time units is $e^{-m(t)}$; the probability of the first occurring at $\tau$ is

$$e^{-\int_\tau^t r(\sigma)\,d\sigma}\, r(\tau)\, d\tau \;.$$

If no distinction is made between random and terminal demand units, the allocation strategy can be chosen to maximize expected number of units delivered to terminal demand. Let $E(n,t)$ represent the expecte number of units delivered given 1) that at time $t$ there are $n$ units remaining and 2) that the number of units allotted to a random demand will maximize the number delivered to terminal demand. Then

$$E(0,t) = 0\;,$$

$$E(1,t) = e^{-m(t)},$$

and, in general,

$$E(n,t) = ne^{-m(t)} + \int_0^t \max_{i=1(1)n-1} \{(1-q^i)E(n-i,\tau)\}e^{-\int_\tau^t r(\sigma)\,d\sigma}\, r(\tau)\, d\tau\;.$$

<u>Theorem 4.2.1.</u> $E(n,t) = e^{-m(t)}\Theta(n, m(t))$ <u>where</u>

$$\Theta(n,x) = n + \int_0^x \max_{i=1(1)n-1} \{(1-q^i)\Theta(n-i,y)\}\, dy$$

$$\Theta(0,x) = 0\;, \quad \Theta(1,x) = 1\;.$$

<u>Proof</u>: The theorem is true for $n = 0, 1, 2$. Assume that it is true for $n < k$. Then

$$\begin{aligned} E(k,t) &= ke^{-m(t)} + \int_0^t \max_{i=1(1)k-1} \{(1-q^i)E(k-i,\tau)\}e^{-\int_\tau^t r(\sigma)\,d\sigma}\, r(\tau)\, d\tau \\ &= ke^{-m(t)} + \int_0^t \max_{i=1(1)k-1} \{(1-q^i)e^{-m(\tau)}\Theta(k-i, m(\tau))\}e^{-\int_\tau^t r(\sigma)\,d\sigma}\, r(\tau)\, d\tau \\ &= e^{-m(t)}\Big[k + \int_0^{m(t)} \max_{i=1(1)k-1} \{(1-q^i)\Theta(k-i,y)\}\, dy\Big] \\ &= e^{-m(t)}\,\Theta(k, m(t))\;. \end{aligned}$$

For $p = q = 0.5$, Table 1 shows $E(n,t)$ for $n = 1(1)10$, $m(t) = 1(1)10$. (Calculations were done using trapezoidal approximate integration with step size 0.05 for $m \leq 5$ and stepsize 0.1 for $m > 5$.) Table 2 for $q = 0.3$ and Table 3 for $q = 0.1$ ($m(t) = 1(1)5$, $n = 1(1)10$, computed using trapezoidal approximate integration with step size 0.1), illustrat the dependence upon $q$.

Table 1

MAXIMUM EXPECTED NUMBER OF UNITS TO TERMINAL DEMAND, $q = 0.5$

| n / m(t) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.00 | .368 | .92 | 1.52 | 2.12 | 2.82 | 3.54 | 4.30 | 5.07 | 5.84 | 6.64 |
| 2.00 | .135 | .41 | .74 | 1.11 | 1.52 | 2.00 | 2.53 | 3.09 | 3.67 | 4.27 |
| 3.00 | .050 | .17 | .35 | .56 | .80 | 1.10 | 1.44 | 1.82 | 2.23 | 2.66 |
| 4.00 | .018 | .07 | .16 | .28 | .42 | .59 | .79 | 1.04 | 1.31 | 1.61 |
| 5.00 | .007 | .03 | .07 | .14 | .21 | .31 | .43 | .58 | .75 | .95 |
| 6.00 | .0025 | .0124 | .0335 | .0657 | .1072 | .159 | .227 | .313 | .418 | .543 |
| 7.00 | .0009 | .0050 | .0147 | .0309 | .0532 | .081 | .118 | .167 | .229 | .304 |
| 8.00 | .0003 | .0020 | .0064 | .0143 | .0260 | .041 | .061 | .088 | .123 | .167 |
| 9.00 | .0001 | .0008 | .0027 | .0065 | .0125 | .021 | .031 | .046 | .065 | .091 |
| 10.00 | .0000 | .0003 | .0012 | .0029 | .0059 | .010 | .016 | .024 | .034 | .048 |

Table 2

MAXIMUM EXPECTED NUMBER OF UNITS TO TERMINAL DEMAND, $q = 0.3$

| n<br>m(t) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.00 | .368 | .98 | 1.67 | 2.39 | 3.10 | 3.85 | 4.67 | 5.51 | 6.37 | 7.23 |
| 2.00 | .135 | .46 | .90 | 1.41 | 1.93 | 2.47 | 3.09 | 3.75 | 4.47 | 5.22 |
| 3.00 | .050 | .20 | .46 | .79 | 1.16 | 1.55 | 1.98 | 2.48 | 3.03 | 3.63 |
| 4.00 | .018 | .09 | .23 | .43 | .68 | .95 | 1.25 | 1.59 | 2.00 | 2.45 |
| 5.00 | .007 | .04 | .11 | .22 | .38 | .57 | .78 | 1.01 | 1.29 | 1.61 |

Table 3

MAXIMUM EXPECTED NUMBER OF UNITS TO TERMINAL DEMAND, $q = 0.1$

| n<br>m(t) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.00 | .368 | 1.05 | 1.87 | 2.72 | 3.59 | 4.45 | 5.32 | 6.19 | 7.05 | 7.92 |
| 2.00 | .135 | .51 | 1.09 | 1.79 | 2.55 | 3.33 | 4.11 | 4.89 | 5.68 | 6.46 |
| 3.00 | .050 | .23 | .59 | 1.10 | 1.72 | 2.39 | 3.09 | 3.79 | 4.50 | 5.21 |
| 4.00 | .018 | .10 | .30 | .64 | 1.10 | 1.64 | 2.24 | 2.86 | 3.50 | 4.14 |
| 5.00 | .007 | .04 | .15 | .35 | .66 | 1.08 | 1.56 | 2.10 | 2.66 | 3.23 |

4. 2. 2. Comparison with Two Distinct Units. A detailed comparison of the expected number of units delivered, (1) when two distinct types of units are used, with (2) when a dual-purpose unit is employed, can be made using numerical tables for P and E. As Table 4 shows, a dual-purpose unit yields a higher expected number of units delivered to terminal demand. We define the effectiveness ratio, E. R., as the maximum expected number to terminal demand obtainable with two types of units divided by that with dual-purpose units. (Both maxima were obtained as described above.)

4. 2. 3. A Suboptimal Strategy. A strategy for dual-purpose units which does not depend on $m(t)$ is given below; we call this the "static strategy". The number of units, $i(n)$, allocated to a demand encountered when $n$ units are available is:

$$i(n) = \text{first } \left( i = 1(1)n : \frac{1-q^{i+1}}{1-q^{i}} < \frac{n-i}{n-i-1} \right) .$$

That is, the static strategy provides the smallest number of units which must be allotted to maximize the product of the probability that the demand is met $(1-q^i)$ times the remaining inventory $(n-i)$.

The expected number of units delivered to terminal demand using the static strategy, $E^*(n,x)$, was computed as follows. The probability $Q(j)$ of exactly $j$ demands arriving in an interval of length $[0, x = m(t)]$ is, from the Poisson model, $Q(j) = e^{-x} x^j / j!$; under the static strategy, "allot $i(k)$ units" $(k = 1, \ldots, j)$, the probability $P(j)$ of successfully meeting these $j$ demands is $P(j) = \prod_{k=1}^{j} (1-q^{i(k)})$, $P(0) = 1$; and the inventory remaining, $n(k)$, is $n(k) = n(k-1) - i(k)$ $(k = 1, \ldots, j,$ $n(0) = n)$. Combining these expressions with the total number of encounters, $J$, for which inventory remains $(n(J) = 1)$ using the static strategy, we obtain:

$$E^*(n,x) = \sum_{j=1}^{J} n(j) P(j) Q(j) .$$

Some typical values of $E^*(n,x)$ for $q = 0.5$ with $n$ betweeen 4 and 40 and $x = m(t) = 1(1)10$ are given in Table 5. The values are in very close numerical agreement with the maximum expected number delivered of Sec. 4. 2. 1 (for example, $n = 8$, $m = 8$: 0.075 versus 0.088; $m = 3$, $n = 10$: 2.57 versus 2.66). Large percentage differences occur only where the actual values are themselves very small (for example, $n = 10$, $m = 10$: 0.036 versus 0.048).

The preceding indicates that a considerable simplification is possible in allocation strategy without appreciable reduction in effectiveness if dual-purpose units are used. No similar simplification now seems possible if simgle-purpose units are used.

4. 3. Extending the Number of Demands Met. In certain situations it might be desirable to use inventory to fill as many demands as possible. In this section our allocation function uses the available

Table 4

SOME VALUES OF THE EFFECTIVENESS RATIO

| Total Units L | Encounter Parameter m( t) | Effectiveness Ratio† by Probability of failure | | |
|---|---|---|---|---|
| | | E. R. (q = .1) | E. R. (q=.5) | E. R. (q=.9) |
| 5 | 1.0 | .75 | .78 | .79 |
| | 5.0 | .63 | .68 | . |
| 10 | 1.0 | .83 | .78 | .81 |
| | 5.0 | .65 | .64 | .67 |
| | 10.00 | .55 | . | |
| 15 | 1.0 | .86 | .79 | .79 |
| | 5.0 | .74 | .67 | .68 |
| | 10.0 | .62 | .63 | |
| | 15.0 | | | |
| 20 | 1.0 | .87 | .80 | .78 |
| | 5.0 | .77 | .67 | .66 |
| | 10.0 | .71 | .63 | |
| | 15.0 | .61 | | |
| | 20.0 | .54 | | |
| 25 | 1.0 | .88 | .82 | .78 |
| | 5.0 | .78 | .68 | .57 |
| | 10.0 | .75 | .64 | |
| | 15.0 | .70 | | |
| | 20.0 | .62 | | |
| 30 | 1.0 | .89 | .83 | .78 |
| | 5.0 | .79 | .70 | .67 |
| | 10.0 | .75 | .64 | |
| | 15.0 | .73 | .61 | |
| | 20.0 | .69 | | |

†Maximum expected number to terminal demand with two types of unit divided by that with dual-purpose units.

Table 5

EXPECTED NUMBER OF UNITS FOR TERMINAL DEMAND UNDER STATIC STRATEGY, $q = 0.5$

| n \ x=m(t) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 2.092 | 1.049 | .507 | .238 | .109 | .049 | .022 | .009 | .004 | .002 |
| 8 | 5.067 | 3.081 | 1.799 | 1.012 | .551 | .291 | .150 | .075 | .037 | .018 |
| 10 | 6.634 | 4.215 | 2.574 | 1.517 | .865 | .480 | .259 | .137 | .071 | .036 |
| 18 | 13.520 | 9.825 | 6.904 | 4.691 | 3.084 | 1.965 | 1.215 | .731 | .428 | .245 |
| 24 | 19.040 | 14.720 | 11.070 | 8.086 | 5.736 | 3.952 | 2.646 | 1.723 | 1.093 | .677 |
| 32 | 26.540 | 21.640 | 17.290 | 13.510 | 10.310 | 7.670 | 5.563 | 3.933 | 2.712 | 1.824 |
| 40 | 34.100 | 28.630 | 23.650 | 19.180 | 15.240 | 11.850 | 9.005 | 6.682 | 4.842 | 3.426 |

units to maximize the expected number of demands met. The maximum expected number of demands met with $t$ units of time and $n$ units of inventory remaining will be denoted by $D(t, n)$.

The expected number of demands met if one is encountered at time $\tau$, and $i$ of $n$ available units are allotted to it is

$$1 \cdot \text{prob}\{\text{not all } i \text{ units fail}\}$$
$$+ \text{prob}\{\text{not all } i \text{ units fail}\} \cdot D(\tau, n-i)$$

Recalling that this probability is $(1-q^i)$, and that the probability of th first encounter at $\tau$ is

$$r(\tau)\, e^{-\int_\tau^t r(\sigma)\, d\sigma}\, d\tau ,$$

we see that

$$D(t, n) = \int_0^t \max_{i=1(1)n} \{(1-q^i)[1 + D(\tau, n-i)]\} \cdot e^{-\int_\tau^t r(\sigma)\, d\sigma}\, r(\tau)\, d\tau .$$

Of course, $D(t, 0) = 0$. $D(t, 1) = p[1-e^{-m(t)}]$, since the probability that one demand is met is the probability $p$ times the probability $(1- e^{-m(t)})$ that at least one demand occurs during the $t$ time units.

Theorem 4.3.1. $D(t, n) = e^{-m(t)}\Omega(m(t), n)$ where

$$\Omega(x, n) = \int_0^x \max_{i=1(1)n} \{(1 - q^i)[e^y + \Omega(y, n-i)]\}\, dy ,$$

$$\Omega(x, 0) = 0$$

$$\Omega(x, 1) = p(e^x-1) .$$

Proof: Assume that the theorem is true for $n < k$. Then

$$D(t, k) = \int_0^t \max_{i=1(1)k} \{(1-q^i)[1+D(\tau, k-i)]\} e^{-\int_\tau^t r(\sigma)\, d\sigma}\, r(\tau)\, d\tau$$

$$= \int_0^t \max_{i=1(1)k} \{(1-q^i)[1 + e^{-m(\tau)}\Omega(m(\tau), k-i)]\} \cdot e^{-\int_\tau^t r(\sigma)\, d\sigma}\, r(\tau)\, d$$

$$= e^{-m(t)} \int_0^{-m(t)} \max_{i=1(1)k} \{(1-q^i)[e^y + \Omega(y, k-i)]\}\, dy$$

$$= e^{-m(t)}\, \Omega(m(t), k) .$$

An exemplary table of $D(t, n)$ for $q = 0.5$, $n = 1(1)10$, and $m(t) = 1(1)10.0$ follows (Table 6 was computed by trapezoidal integration with step size $0.05$ for $m(t) \leq 5$ and $0.1$ otherwise).

The maximum expected number of demands met does not depend on either time or the expected number of encounters $(x = m(t))$ providec either is sufficiently large. This is true because each of the limited numbers of units can fill only one demand. Hence after a point, an increase in the actual number of encounters does not increase the maximum expected number of demands met. This saturation effect can be seen in Table 6 ; each of the $D(t, n)$ approach a limiting value as $m(t)$ increases. To begin an analysis of this property we make the following definition:

Definition 4.3.1. $D(n) = \lim_{t \to \infty} D(t, n)$. It is clear that

$$D(0) = 0 ,$$

$$D(1) = p = 1-q ,$$

since with zero or one unit the expected number of demands met is, respectively, none and one times the probability of success of a unit. The following lemma gives $D(n)$ for higher values of $n$.

Lemma 4.3.1.

$$D(n) = \max_{i=1(1)n} \{(1-q^i)[1 + D(n-i)]\} ,$$

$$D(0) = 0 , \quad D(1) = 1-q .$$

Proof: By definition,

$$D(n) = \lim_{x \to \infty} e^{-x}\Omega(x, n)$$

Since

$$\Omega(x, n) = \int_0^x \max_{i=1(1)n} \{(1-q^i)[e^y + \Omega(y, n-i)]\} \, dy ,$$

$$\Omega(x, n) > ke^x , \quad k > 0 ;$$

thus this limit is of the form $\infty/\infty$. Applying L'Hospital's rule,

$$D(n) = \lim_{x \to \infty} \frac{\Omega(x, n)}{e^x} = \lim_{x \to \infty} \frac{\Omega'(x, n)}{e^x}$$

$$= \lim_{x \to \infty} \frac{\max_{i=1(1)n} \{(1-q^i)[e^x + \Omega(x, n-i)]\}}{e^x}$$

$$= \lim_{x \to \infty} \max_{i=1(1)n} \{(1-q^i)[1 + e^{-x}\Omega(x, n-i)]\}$$

$$= \max_{i=1(1)n} \{(1-q^i)[1 + \lim_{x \to \infty} e^{-x} \Omega(x, n-i)]\} .$$

Lemma 4.3.1 was used to calculate $D(n)$, the limit of the maximum expected number of demands met as $m(t)$ (and hence the number of encounters) increased without bound. The values are given in Table 7 for $n = 1(1)10$, $q = 0.1(0.1)0.9$. The values of $D(n)$ for $q = 0.5$ correspond to the limiting values we previously presented (Table 6).

Table 6

MAXIMUM EXPECTED NUMBER OF DEMANDS MET, $q = 0.5$

| n / m(t) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.00 | .32 | .47 | .58 | .67 | .73 | .78 | .82 | .85 | .87 | .89 |
| 2.00 | .43 | .65 | .88 | 1.02 | 1.15 | 1.27 | 1.35 | 1.43 | 1.50 | 1.56 |
| 3.00 | .48 | .71 | 1.01 | 1.18 | 1.36 | 1.55 | 1.67 | 1.79 | 1.91 | 2.03 |
| 4.00 | .49 | .74 | 1.08 | 1.26 | 1.47 | 1.71 | 1.85 | 2.00 | 2.17 | 2.32 |
| 5.00 | .50 | .75 | 1.11 | 1.29 | 1.54 | 1.79 | 1.94 | 2.13 | 2.32 | 2.49 |
| 6.00 | .50 | .75 | 1.12 | 1.30 | 1.57 | 1.83 | 1.99 | 2.20 | 2.41 | 2.58 |
| 7.00 | .50 | .75 | 1.12 | 1.31 | 1.58 | 1.85 | 2.01 | 2.24 | 2.46 | 2.63 |
| 8.00 | .50 | .75 | 1.12 | 1.31 | 1.59 | 1.85 | 2.02 | 2.25 | 2.48 | 2.66 |
| 9.00 | .50 | .75 | 1.13 | 1.31 | 1.59 | 1.86 | 2.02 | 2.26 | 2.50 | 2.68 |
| 10.00 | .50 | .75 | 1.13 | 1.31 | 1.59 | 1.86 | 2.02 | 2.27 | 2.50 | 2.68 |

Table 7

LIMIT OF MAXIMUM EXPECTED NUMBER OF DEMANDS MET AS m(t) INCREASES VERSUS NUMBER OF UNITS INITIALLY AVAILABLE, n, AND UNIT FAILURE PROBABILITY, $q = 1 - p$

| q = <br> n | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | .900 | .800 | .700 | .600 | .500 | .400 | .300 | .200 | .100 |
| 2 | 1.710 | 1.440 | 1.190 | .960 | .750 | .640 | .510 | .360 | .190 |
| 3 | 2.439 | 1.952 | 1.547 | 1.344 | 1.125 | .896 | .663 | .488 | .271 |
| 4 | 3.095 | 2.362 | 1.992 | 1.646 | 1.313 | 1.098 | .854 | .590 | .344 |
| 5 | 3.686 | 2.834 | 2.318 | 1.969 | 1.594 | 1.286 | .992 | .708 | .410 |
| 6 | 4.217 | 3.227 | 2.724 | 2.223 | 1.859 | 1.486 | 1.147 | .807 | .469 |
| 7 | 4.695 | 3.681 | 3.019 | 2.494 | 2.023 | 1.650 | 1.264 | .914 | .522 |
| 8 | 5.165 | 4.058 | 3.388 | 2.779 | 2.270 | 1.826 | 1.409 | 1.003 | .574 |
| 9 | 5.638 | 4.493 | 3.657 | 3.017 | 2.502 | 1.990 | 1.542 | 1.098 | .626 |
| 10 | 6.103 | 4.856 | 3.993 | 3.270 | 2.681 | 2.164 | 1.657 | 1.176 | .678 |

5. Conclusion. The models we have presented made possible quantitative analysis of a complex system. In themselves, they provide examples of stochastic allocation situations with an interesting mathematical structure.

## Appendix

The Function $P(n, t, v)$. We can differentiate the expression for $P(n, t, v)$ on page 16 with respect to $t$ for fixed $n$ and $v$ to obtain the following lemma.

Lemma A. 1.

$$\frac{dP}{dt}(n, t, v) = -r(t)\left[P(n, t, v) - \max_{i=0(1)n} \{(1-vq^i) P(n-i, t, v)\}\right].$$

Proof:

$$\frac{dP}{dt}(n, t, v) = -r(t)\Big[e^{-m(t)} + \int_0^t \max_{i=0(1)n} \{(1-vq^i) P(n-i, \tau, v)\}$$

$$\times\, e^{-\int_\tau^t r(\sigma)\, d\sigma} r(\tau)\, d\tau - \max_{i=0(1)n} (1-vq^i) P(n-i, t, v)\Big].$$

We now relax the maximum condition in the differential equation of Lemma A. 1 and consider the solutions of the equations that result when $i$ is equal to zero and $n$.

Let $P(n, t, v; i)$ be the solution to

$$\frac{dP}{dt}(n, t, v) = -r(t)\left[P(n, t, v) - (1-vq^i) P(n-i, t, v)\right],$$

with initial condition $P(n, \tau, v)$.

Lemma A. 2.

$$P(n, t, v; n) = e^{-m(t)}\left[e^{m(\tau)} P(n, \tau, v) + (e^{wm(t)} - e^{wm(\tau)})(1 + \tfrac{v}{w}(1-q^n))\right]$$

Proof: Using $P(0, t, v) = e^{-vm(t)}$, multiplying through by $e^{m(t)}$, and replacing $P(n, t, v; n)$ with $P(t)$ (or $P$, when no confusion results), for brevity, the differential equation is the same as

$$d[e^{m(t)} P] = (1-vq^n)\, r(t)\, e^{wm(t)}\, dt.$$

Thus,

$$\begin{aligned} e^{m(t)} P(t) - e^{m(\tau)} P(\tau) &= (1-vq^n) \int_\tau^t r(t)\, e^{wm(t)}\, dt \\ &= (1-vq^n) \int_{m(\tau)}^{m(t)} e^{wy}\, dy \\ &= \frac{1-vq^n}{1-v}(e^{wm(t)} - e^{wm(\tau)}), \end{aligned}$$

which reduces to the first expression above. This simplifies when $\tau = 0$, since $m(0) = 0$ and $P(n, 0, v) = 1$.

Lemma A. 3.

$$P(n, t, v; 0) = e^{-v[m(t) - m(\tau)]} \cdot P(n, \tau, v) .$$

Proof: In this case the differential equation is the same as

$$\frac{dP(n, t, v; 0)}{P(n, t, v; 0)} = -r(t)\, v\, dt .$$

We can now prove the following theorem:

Theorem A. 1. If allocation must be via a "bang-bang" strategy (all or none of the available units), all the units should be used for any demand encountered.

Proof: The probabilities of success under each alternative are given by Lemmas A. 2 and A. 3 with $m(0) = 0$, $P(n, 0, v) = 1$. But

$$e^{-vm(t)}[1 + \frac{v}{w}(1-q^n)(1-e^{-wm(t)})] > e^{-vm(t)} ,$$

since $w > 0$, and if $t \neq 0$, $m(t) > 0$; this implies $e^{wm(t)} > 1$, which insures the stated inequality. Hence

$$P(n, t, v; 0) < P(n, t, v; n) ,$$

and the theorem is proven.

Corollary A. 1.

$$P(1, t, v) = e^{-vm(t)}[1 + \frac{v}{w} p(1 - e^{-wm(t)})] .$$

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