

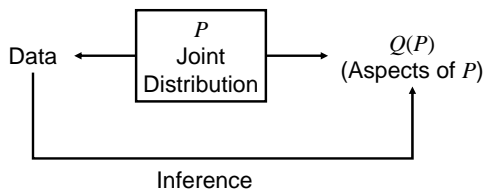
THE MATHEMATICS OF CAUSE AND COUNTERFACTUALS

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OUTLINE

- Inference: Statistical vs. Causal distinctions and mental barriers
- Formal semantics for counterfactuals: definition, axioms, graphical representations
- Inference to three types of claims:
 1. Effect of potential interventions
 2. Attribution (Causes of Effects)
 3. Direct and indirect effects

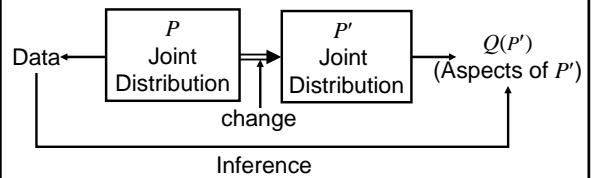
TRADITIONAL STATISTICAL INFERENCE PARADIGM



e.g.,
 Infer whether customers who bought product A would also buy product B.
 $Q = P(B | A)$

FROM STATISTICAL TO CAUSAL ANALYSIS: 1. THE DIFFERENCES

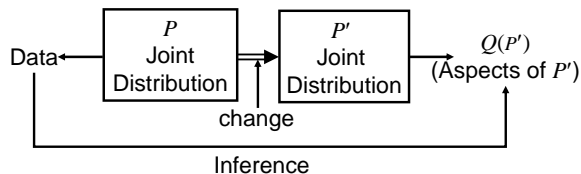
Probability and statistics deal with static relations



What happens when P changes?
 e.g.,
 Infer whether customers who bought product A would still buy A if we were to double the price.

FROM STATISTICAL TO CAUSAL ANALYSIS: 1. THE DIFFERENCES

What remains invariant when P changes say, to satisfy $P'(price=2)=1$



Note: $P'(v) \neq P(v | price = 2)$

P does not tell us how it ought to change
 e.g. Curing symptoms vs. curing diseases
 e.g. Analogy: mechanical deformation

FROM STATISTICAL TO CAUSAL ANALYSIS: 1. THE DIFFERENCES (CONT)

1. Causal and statistical concepts do not mix.

CAUSAL	STATISTICAL
Spurious correlation	Regression
Randomization / Intervention	Association / Independence
Confounding / Effect	"Controlling for" / Conditioning
Instrumental variable	Odd and risk ratios
Strong Exogeneity	Collapsibility / Granger causality
Explanatory variables	Propensity score
- 2.
- 3.
- 4.

FROM STATISTICAL TO CAUSAL ANALYSIS: 2. MENTAL BARRIERS

- Causal and statistical concepts do not mix.

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statistical assumptions + data	} ⇒ causal conclusions
causal assumptions	
- Causal assumptions cannot be expressed in the mathematical language of standard statistics.
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causal assumptions	
- Causal assumptions cannot be expressed in the mathematical language of standard statistics.
- Non-standard mathematics:
 - Structural equation models (Wright, 1920; Simon, 1960)
 - Counterfactuals (Neyman-Rubin (Y_x), Lewis ($x \rightarrow Y$))

WHY CAUSALITY NEEDS SPECIAL MATHEMATICS

Scientific Equations (e.g., Hooke's Law) are non-algebraic
e.g., Length (Y) equals a constant (2) times the weight (X)
Correct notation:

$Y = 2X$	$X = 1$
$X = 1$	$Y = 2$
<u>Process information</u>	<u>The solution</u>

Had X been 3, Y would be 6.
If we raise X to 3, Y would be 6.
Must "wipe out" $X = 1$.

WHY CAUSALITY NEEDS SPECIAL MATHEMATICS

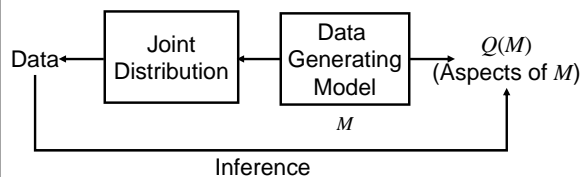
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(or)

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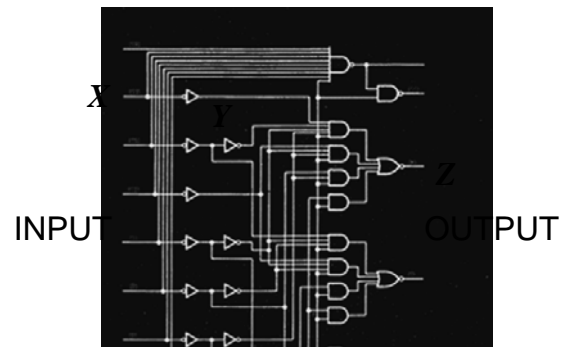
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THE STRUCTURAL MODEL PARADIGM



M – Invariant strategy (mechanism, recipe, law, protocol) by which Nature assigns values to variables in the analysis.

FAMILIAR CAUSAL MODEL ORACLE FOR MANIPULATION



STRUCTURAL CAUSAL MODELS

Definition: A structural causal model is a 4-tuple $\langle V, U, F, P(u) \rangle$, where

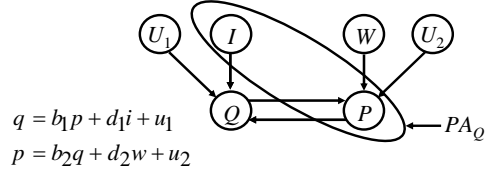
- $V = \{V_1, \dots, V_n\}$ are observable variables
- $U = \{U_1, \dots, U_m\}$ are background variables
- $F = \{f_1, \dots, f_n\}$ are functions determining V ,
 $v_i = f_i(v, u)$
- $P(u)$ is a distribution over U

$P(u)$ and F induce a distribution $P(v)$ over observable variables

STRUCTURAL MODELS AND CAUSAL DIAGRAMS

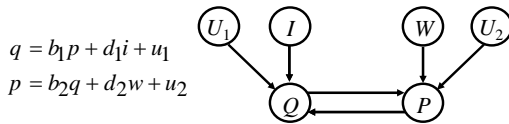
The arguments of the functions $v_i = f_i(v, u)$ define a graph
 $v_i = f_i(pa_i, u_i) \quad PA_i \subseteq V \setminus V_i \quad U_i \subseteq U$

Example: Price – Quantity equations in economics



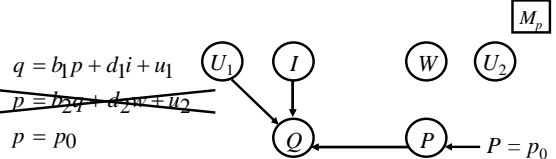
STRUCTURAL MODELS AND INTERVENTION

Let X be a set of variables in V .
 The action $do(x)$ sets X to constants x regardless of the factors which previously determined X .
 $do(x)$ replaces all functions f_i determining X with the constant functions $X=x$, to create a mutilated model M_x .



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CAUSAL MODELS AND COUNTERFACTUALS

Definition:
 The sentence: “ Y would be y (in situation u), had X been x ,” denoted $Y_x(u) = y$, means:
 The solution for Y in a mutilated model M_x (i.e., the equations for X replaced by $X = x$) with input $U = u$, is equal to y .

The Fundamental Equation of Counterfactuals:

$$Y_x(u) = Y_{M_x}(u)$$

CAUSAL MODELS AND COUNTERFACTUALS

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The empirical claim of $y = f_Y(pa_Y, u_Y)$

$$Y_{pa_Y, s}(u) = Y_{pa_Y}(u) \quad \text{for } S \text{ outside } PA_Y$$

The empirical claim of $U_Y \perp\!\!\!\perp U_Z$

$$Y_{pa_Y} \perp\!\!\!\perp Z_{pa_Z}$$

CAUSAL MODELS AND COUNTERFACTUALS

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 The sentence: "Y would be y (in situation u), had X been x," denoted $Y_x(u) = y$, means:
 The solution for Y in a mutilated model M_x (i.e., the equations for X replaced by $X = x$) with input $U = u$, is equal to y.

- Joint probabilities of counterfactuals:

$$P(Y_x = y, Z_w = z) = \sum_{u: Y_x(u)=y, Z_w(u)=z} P(u)$$

In particular:

$$P(y | do(x)) \stackrel{\Delta}{=} P(Y_x = y) = \sum_{u: Y_x(u)=y} P(u)$$

$$PN(Y_{x'} = y' | x, y) = \sum_{u: Y_{x'}(u)=y'} P(u | x, y)$$

REGRESSION VS. STRUCTURAL EQUATIONS (THE CONFUSION OF THE CENTURY)

Regression (claimless, nonfalsifiable):

$$Y = ax + \epsilon_y$$
 Structural (empirical, falsifiable):

$$Y = bx + u_y$$
 Assumptions:

$$E(Y|do(x)) = E(Y|do(x), do(z)) = bx$$
 The mothers of all questions:
 Q. When would b equal a?
 A. When all back-door paths are blocked, or $Y_x \perp\!\!\!\perp X$
 Q. When is b estimable by regression methods?
 A. Graphical criteria available

AXIOMS OF CAUSAL COUNTERFACTUALS

Y would be y, had X been x (in state $U = u$)

- Definiteness**
 $\exists x \in X \text{ s.t. } X_y(u) = x$
- Uniqueness**
 $(X_y(u) = x) \& (X_{y'}(u) = x') \Rightarrow x = x'$
- Effectiveness**
 $X_{xw}(u) = x$
- Composition**
 $W_x(u) = w \Rightarrow Y_{xw}(u) = Y_x(u)$
- Reversibility**
 $(Y_{xw}(u) = y) \& (W_{xy}(u) = w) \Rightarrow Y_x(u) = y$

INFERRING THE EFFECT OF INTERVENTIONS

The problem:
 To predict the impact of a proposed intervention using data obtained prior to the intervention.

The solution (conditional):
 Causal Assumptions + Data \rightarrow Policy Claims

- Mathematical tools for communicating causal assumptions formally and transparently.
- Deciding (mathematically) whether the assumptions communicated are sufficient for obtaining consistent estimates of the prediction required.
- ~~Deciding (mathematically) whether the assumptions communicated are sufficient for obtaining consistent estimates of the prediction required.~~
 a causal model represents a prediction that, if performed, would render a consistent estimate feasible.

NON-PARAMETRIC STRUCTURAL MODELS

Given $P(x,y,z)$, should we ban smoking?

Linear Analysis
 $x = u_1,$
 $z = \alpha x + u_2,$
 $y = \beta z + \gamma u_1 + u_3.$
 Find: $\alpha \cdot \beta$

Nonparametric Analysis
 $x = f_1(u_1),$
 $z = f_2(x, u_2),$
 $y = f_3(z, u_1, u_3).$
 Find: $P(y|do(x))$

EFFECT OF INTERVENTION AN EXAMPLE

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 $z = \alpha x + u_2,$
 $y = \beta z + \gamma u_1 + u_3.$
 Find: $\alpha \cdot \beta$

Nonparametric Analysis
 $x = const.$
 $z = f_2(x, u_2),$
 $y = f_3(z, u_1, u_3).$
 Find: $P(y|do(x)) \stackrel{\Delta}{=} P(Y=y)$ in new model

IDENTIFIABILITY

Definition:

Let $Q(M)$ be any quantity defined on a causal model M , and let A be a set of assumption.

Q is identifiable relative to A iff

$$P(M_1) = P(M_2) \Rightarrow Q(M_1) = Q(M_2)$$

for all M_1, M_2 , that satisfy A .

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In other words, Q can be determined uniquely from the probability distribution $P(V)$ of the endogenous variables, V , and assumptions A .

-

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In this talk:

A: Assumptions encoded in the diagram

Q_1 : $P(y|do(x))$ Causal Effect (= $P(Y_x=y)$)

Q_2 : $P(Y_x=y' | x, y)$ Probability of necessity

Q_3 : $E(Y_{XZ_x'})$ Direct Effect Q_4 : $P(Y_x=y | x)$ ETT

○ THE FUNDAMENTAL THEOREM OF CAUSAL INFERENCE

Causal Markov Theorem:

Any distribution generated by Markovian structural model M (recursive, with independent disturbances) can be factorized as

$$P(v_1, v_2, \dots, v_n) = \prod_i P(v_i | pa_i)$$

Where pa_i are the (values of) the parents of V_i in the causal diagram associated with M .

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Corollary-1: (Truncated factorization, Manipulation Theorem)

The distribution generated by an intervention $do(X=x)$ (in a Markovian model M) is given by the truncated factorization

$$P(v_1, v_2, \dots, v_n | do(x)) = \prod_{i|V_i \notin X} P(v_i | pa_i) \Big|_{X=x}$$

(G-estimation)

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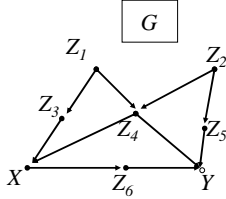
Corollary-2: (Parents adjustment formula)

The causal effect of X on Y , $P(Y=y | do(X=x))$ (in a Markovian model M) is given by

$$P(y | do(x)) = \sum_{pa_X} P(y | x, pa_X) P(pa_X)$$

EFFECT OF INTERVENTION THE GENERAL CASE

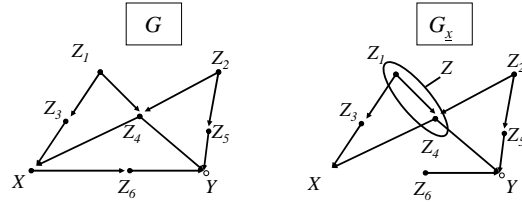
Find the effect of X on Y , $P(y|do(x))$, given the causal assumptions shown in G , where Z_1, \dots, Z_k are auxiliary variables.



Can $P(y|do(x))$ be estimated if only a subset, Z , can be measured?

ELIMINATING CONFOUNDING BIAS THE BACK-DOOR CRITERION

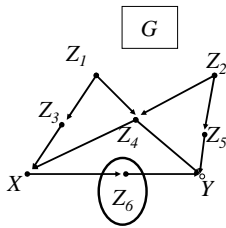
$P(y | do(x))$ is estimable if there is a set Z of variables such that Z d -separates X from Y in $G_{\bar{X}}$.



Moreover, $P(y | do(x)) = \sum_z P(y | x, z) P(z)$ ("adjusting" for Z)

EFFECT OF INTERVENTION BEYOND ADJUSTMENT

Theorem (Tian-Pearl 2002)
We can identify $P(y|do(x))$ if there is no child Z of X connected to X by a confounding path.



INFERENCE ACROSS DESIGNS

Problem:

Predict $P(y | do(x))$ from a study in which only Z can be controlled.

Solution:

Determine if $P(y | do(x))$ can be reduced to a mathematical expression involving only $do(z)$.

○ RULES OF CAUSAL CALCULUS

Rule 1: Ignoring observations

$$P(y | do\{x\}, z, w) = P(y | do\{x\}, w) \text{ if } (Y \perp\!\!\!\perp Z | X, W)_{G_{\bar{X}}}$$

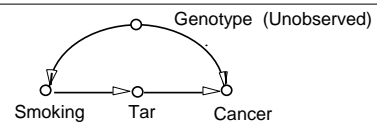
Rule 2: Action/observation exchange

$$P(y | do\{x\}, do\{z\}, w) = P(y | do\{x\}, z, w) \text{ if } (Y \perp\!\!\!\perp Z | X, W)_{G_{\bar{X}\bar{Z}}}$$

Rule 3: Ignoring actions

$$P(y | do\{x\}, do\{z\}, w) = P(y | do\{x\}, w) \text{ if } (Y \perp\!\!\!\perp Z | X, W)_{G_{\bar{X}\bar{Z}(W)}}$$

○ DERIVATION IN CAUSAL CALCULUS

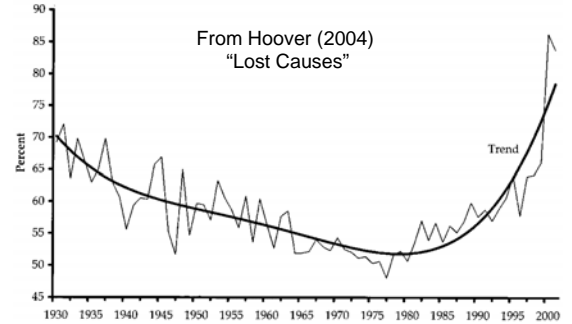


$$\begin{aligned} P(c | do\{s\}) &= \sum_t P(c | do\{s\}, t) P(t | do\{s\}) && \text{Probability Axioms} \\ &= \sum_t P(c | do\{s\}, do\{t\}) P(t | do\{s\}) && \text{Rule 2} \\ &= \sum_t P(c | do\{s\}, do\{t\}) P(t | s) && \text{Rule 2} \\ &= \sum_t P(c | do\{t\}) P(t | s) && \text{Rule 3} \\ &= \sum_{s'} \sum_t P(c | do\{t\}, s') P(s' | do\{t\}) P(t | s) && \text{Probability Axioms} \\ &= \sum_{s'} \sum_t P(c | t, s') P(s' | do\{t\}) P(t | s) && \text{Rule 2} \\ &= \sum_{s'} \sum_t P(c | t, s') P(s') P(t | s) && \text{Rule 3} \end{aligned}$$

EFFECT OF INTERVENTION COMPLETE IDENTIFICATION

- Complete calculus for reducing $P(y|do(x), z)$ to expressions void of do -operators.
- Complete graphical criterion for identifying causal effects (Shpitser and Pearl, 2006).
- Complete graphical criterion for empirical testability of counterfactuals (Shpitser and Pearl, 2007).

THE CAUSAL RENAISSANCE: VOCABULARY IN ECONOMICS



THE CAUSAL RENAISSANCE: USEFUL RESULTS

1. Complete formal semantics of counterfactuals
2. Transparent language for expressing assumptions
3. Complete solution to causal-effect identification
4. Legal responsibility (bounds)
5. Imperfect experiments (universal bounds for IV)
6. Integration of data from diverse sources
7. **Direct and Indirect effects,**
8. Complete criterion for counterfactual testability

COUNTERFACTUALS AT WORK ETT

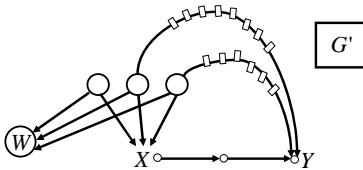
1. I took a pill to fall asleep. Should I have?
2. What would terminating a program do to those enrolled?

$$ETT = P(Y_x = y | x)$$

ETT - IDENTIFICATION

Theorem (Shpitser-Pearl, 2009)

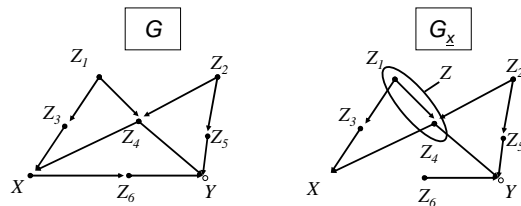
ETT is identifiable in G iff $P(y | do(x), w)$ is identifiable in G'



Moreover, $ETT = P(y | do(x), w) |_{G' w=x'}$

ETT - THE BACK-DOOR CRITERION

$P(Y_x = y | x')$ is identifiable in G if there is a set Z of variables such that Z d -separates X from Y in G_x .



Moreover, $P(y | do(x)) = \sum_z P(y | x, z) P(z | x')$
("adjusting" for Z)

DETERMINING THE CAUSES OF EFFECTS (The Attribution Problem)

- Your Honor! My client (Mr. A) died BECAUSE he used that drug.



-

DETERMINING THE CAUSES OF EFFECTS (The Attribution Problem)

- Your Honor! My client (Mr. A) died BECAUSE he used that drug.



- Court to decide if it is MORE PROBABLE THAN NOT that A would be alive BUT FOR the drug!
 $PN = P(? | A \text{ is dead, took the drug}) \geq 0.50$

THE PROBLEM

Semantical Problem:

1. What is the meaning of $PN(x,y)$:
"Probability that event y would not have occurred if it were not for event x , given that x and y did in fact occur."

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Answer:

$$PN(x, y) = P(Y_{x'} = y' | x, y)$$

Computable from M

THE PROBLEM

Semantical Problem:

1. What is the meaning of $PN(x,y)$:
"Probability that event y would not have occurred if it were not for event x , given that x and y did in fact occur."

Analytical Problem:

2. Under what condition can $PN(x,y)$ be learned from statistical data, i.e., observational, experimental and combined.

TYPICAL THEOREMS

(Tian and Pearl, 2000)

- Bounds given combined nonexperimental and experimental data

$$\max \left\{ \frac{0}{P(x,y)}, \frac{P(y) - P(y_{x'})}{P(x,y)} \right\} \leq PN \leq \min \left\{ \frac{1}{P(x,y)}, \frac{P(y_{x'})}{P(x,y)} \right\}$$

- Identifiability under monotonicity (Combined data)

$$PN = \frac{P(y|x) - P(y|x')}{P(y|x)} + \frac{P(y|x') - P(y_{x'})}{P(x,y)}$$

corrected Excess-Risk-Ratio

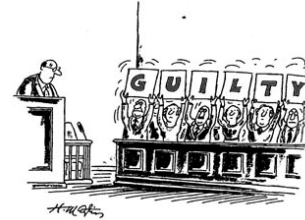
CAN FREQUENCY DATA DECIDE LEGAL RESPONSIBILITY?

	Experimental		Nonexperimental	
	$do(x)$	$do(x')$	x	x'
Deaths (y)	16	14	2	28
Survivals (y')	984	986	998	972
	1,000	1,000	1,000	1,000

- Nonexperimental data: drug usage predicts longer life
- Experimental data: drug has negligible effect on survival
- Plaintiff: Mr. A is special.
 1. He actually died
 2. He used the drug by choice
- Court to decide (given both data):
Is it more probable than not that A would be alive but for the drug?

$$PN \triangleq P(Y_{x'} = y' | x, y) > 0.50$$

SOLUTION TO THE ATTRIBUTION PROBLEM



- WITH PROBABILITY ONE $1 \leq P(y'_{x'} | x, y) \leq 1$
- Combined data tell more than each study alone

EFFECT DECOMPOSITION (direct vs. indirect effects)

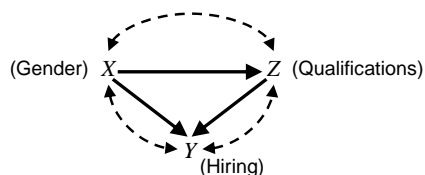
1. Why decompose effects?
2. What is the semantics of direct and indirect effects?
3. What are the policy implications of direct and indirect effects?
4. When can direct and indirect effect be estimated consistently from experimental and nonexperimental data?

WHY DECOMPOSE EFFECTS?

1. To understand how Nature works
2. To comply with legal requirements
3. To predict the effects of new type of interventions:
Signal routing, rather than variable fixing

LEGAL IMPLICATIONS OF DIRECT EFFECT

Can data prove an employer guilty of hiring discrimination?



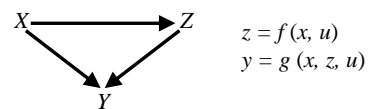
What is the direct effect of X on Y ?

$$E(Y | do(x_1), do(z)) - E(Y | do(x_0), do(z))$$

(averaged over z) Adjust for Z ? No! No!

NATURAL SEMANTICS OF AVERAGE DIRECT EFFECTS

Robins and Greenland (1992) – “Pure”



Average Direct Effect of X on Y : $DE(x_0, x_1; Y)$

The expected change in Y , when we change X from x_0 to x_1 , and, for each u , we keep Z constant at whatever value it attained before the change.

$$E[Y_{x_1 Z_{x_0}} - Y_{x_0}]$$

In linear models, $DE =$ Controlled Direct Effect

SEMANTICS AND IDENTIFICATION OF NESTED COUNTERFACTUALS

Consider the quantity $Q \triangleq E_u[Y_{xZ_{x^*}}(u)]$

Given $\langle M, P(u) \rangle$, Q is well defined

Given u , $Z_{x^*}(u)$ is the solution for Z in M_{x^*} , call it z

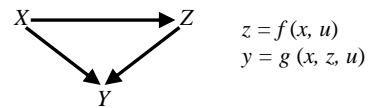
$Y_{xZ_{x^*}}(u)$ is the solution for Y in M_{xz}

Can Q be estimated from $\left\{ \begin{array}{l} \text{experimental} \\ \text{nonexperimental} \end{array} \right\}$ data?

Experimental: nest-free expression

Nonexperimental: subscript-free expression

NATURAL SEMANTICS OF INDIRECT EFFECTS



Indirect Effect of X on Y : $IE(x_0, x_1; Y)$

The expected change in Y when we keep X constant, say at x_0 , and let Z change to whatever value it would have attained had X changed to x_1 .

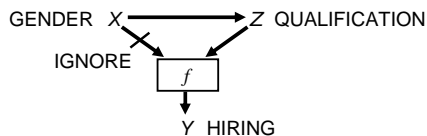
$$E[Y_{x_0 Z_{x_1}} - Y_{x_0}]$$

In linear models, $IE = TE - DE$

POLICY IMPLICATIONS OF INDIRECT EFFECTS

What is the indirect effect of X on Y ?

The effect of Gender on Hiring if sex discrimination is eliminated.



Blocking a link – a new type of intervention

RELATIONS BETWEEN TOTAL, DIRECT, AND INDIRECT EFFECTS

Theorem 5: The total, direct and indirect effects obey The following equality

$$TE(x, x^*; Y) = DE(x, x^*; Y) - IE(x^*, x; Y)$$

In words, the total effect (on Y) associated with the transition from x^* to x is equal to the difference between the direct effect associated with this transition and the indirect effect associated with the reverse transition, from x to x^* .

EXPERIMENTAL IDENTIFICATION OF AVERAGE DIRECT EFFECTS

Theorem: If there exists a set W such that

$$Y_{xz} \perp\!\!\!\perp Z_{x^*} \mid W \text{ for all } z \text{ and } x$$

Then the average direct effect

$$DE(x, x^*; Y) = E(Y_{xZ_{x^*}}) - E(Y_{x^*Z})$$

Is identifiable from experimental data and is given by

$$DE(x, x^*; Y) = \sum_{w, z} [E(Y_{xz} \mid w) - E(Y_{x^*z} \mid w)] P(Z_{x^*} = z \mid w) P(w)$$

GRAPHICAL CONDITION FOR EXPERIMENTAL IDENTIFICATION OF DIRECT EFFECTS

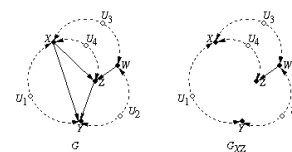
Theorem: If there exists a set W such that

$$(Y \perp\!\!\!\perp Z \mid W)_{G_{XZ}} \text{ and } W \subseteq ND(X \cup Z)$$

then,

$$DE(x, x^*; Y) = \sum_{w, z} [E(Y_{xz} \mid w) - E(Y_{x^*z} \mid w)] P(Z_{x^*} = z \mid w) P(w)$$

Example:



GENERAL PATH-SPECIFIC EFFECTS (Def.)

Form a new model, M_g^* , specific to active subgraph g
 $f_i^*(pa_i, u; g) = f_i(pa_i(g), pa_i(\bar{g}), u)$
 Definition: g -specific effect
 $E_g(x, x^*; Y)_M = TE(x, x^*; Y)_{M_g^*}$
 Nonidentifiable even in Markovian models

SUMMARY OF RESULTS

1. Formal semantics of path-specific effects, based on signal blocking, instead of value fixing.
2. Path-analytic techniques extended to nonlinear and nonparametric models.
3. Meaningful (graphical) conditions for estimating direct and indirect effects from experimental and nonexperimental data.

CONCLUSIONS

Structural-model semantics, enriched with logic and graphs, provides:

- Complete formal basis for causal and counterfactual reasoning
- Unifies the graphical, potential-outcome and structural equation approaches
- Provides friendly and formal solutions to century-old problems and confusions.

TWO PARADIGMS FOR CAUSAL INFERENCE

Observed: $P(X, Y, Z, \dots)$
 Conclusions needed: $P(Y_x = y), P(X_y = x | Z = z) \dots$

How do we connect observables, X, Y, Z, \dots to counterfactuals Y_x, X_z, Z_y, \dots ?

<u>N-R model</u>	<u>Structural model</u>
Counterfactuals are primitives, new variables	Counterfactuals are derived quantities
Super-distribution $P^*(X, Y, \dots, Y_x, X_z, \dots)$	Subscripts modify the model and distribution
X, Y, Z constrain Y_x, X_z, \dots	$P(Y_x = y) = P_{M_x}(Y = y)$

"SUPER" DISTRIBUTION IN N-R MODEL

X	Y	Z	$Y_{x=0}$	$Y_{x=1}$	$X_{z=0}$	$X_{z=1}$	$X_{y=0} \dots$	U
0	0	0	0	1	0	0	0...	u_1
0	1	1	1	0	1	0	1...	u_2
0	0	0	1	0	0	1	1...	u_3
1	0	0	1	0	0	1	0...	u_4

inconsistency: $x = 0 \Rightarrow Y_{x=0} = Y$ $Y = xY_1 + (1-x)Y_0$

Defines: $P^*(X, Y, Z, \dots, Y_x, Z_y, \dots, Y_{xz}, Z_{xy}, \dots)$
 $P^*(Y_x = y | Z, X_z)$
 $Y_x \perp\!\!\!\perp X | Z_y$

TYPICAL INFERENCE IN N-R MODEL

Find $P^*(Y_x = y)$ given covariate Z ,

$$P^*(Y_x = y) = \sum_z P^*(Y_x = y | z) P(z)$$

Assume ignorability: $Y_x \perp\!\!\!\perp X | Z$
 $\Rightarrow \sum_z P^*(Y_x = y | x, z) P(z)$

Assume consistency: $X = x \Rightarrow Y_x = Y$
 $\Rightarrow \sum_z P(y | x, z) P(z)$

?

Try it: $X \rightarrow Y \rightarrow Z$

Problems:

- 1) $Y_x \perp\!\!\!\perp X | Z$ judgmental & opaque
- 2) Is consistency the only connection between X, Y and Y_x ?

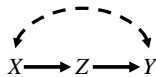
DIFFICULTIES WITH ALGEBRAIC LANGUAGE:

Consider a set of assumptions:

$$\begin{aligned} Z_x(u) &= Z_{yx}(u), \\ X_y(u) &= X_{zy}(u) = X_z(u) = X(u), \\ Y_z(u) &= Y_{zx}(u), \\ Z_x &\perp\!\!\!\perp \{Y_z, X\} \end{aligned}$$

Unfriendly:
Consistent?, complete?, redundant?, arguable?

Friendly language:



GRAPHICAL – COUNTERFACTUALS SYMBIOSIS

Every causal graph expresses counterfactual assumptions, e.g., $X \rightarrow Y \rightarrow Z$

1. Missing arrows $Y \leftarrow Z$ $Y_{x,z}(u) = Y_x(u)$

2. Missing arcs $Y \rightleftarrows Z$ $Y_x \perp\!\!\!\perp Z_y$

consistent, and readable from the graph.

Every theorem in SCM is a theorem in Potential-Outcome Model, and conversely.

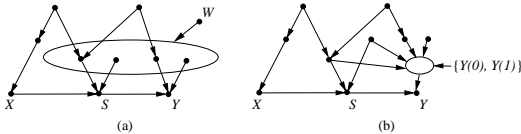
DEMYSTIFYING STRONG IGNORABILITY

$$\{Y(0), Y(1)\} \perp\!\!\!\perp X \mid Z \quad (\text{SI})$$

$$P(y \mid do(x)) = \sum_z P(y \mid z, x) P(z) \quad (\text{Z-admissible})$$

$$(X \perp\!\!\!\perp Y \mid Z)_{G_{\underline{X}}} \quad (\text{Back-door})$$

Is there a W in G such that $(W \perp\!\!\!\perp X \mid Z)_{G_{\underline{X}}} \Rightarrow \text{SI}$?



WHAT PROPENSITY SCORE (PS) PRACTITIONERS NEED TO KNOW

$$L(s) = P(X = 1 \mid S = s) \quad X \perp\!\!\!\perp S \mid L(s)$$

$$\sum_s P(y \mid s, x) P(s) = \sum_t P(y \mid l, x) P(l)$$

1. The asymptotic bias of PS is EQUAL to that of ordinary adjustment (for same S).
2. Including an additional covariate in the analysis CAN SPOIL the bias-reduction potential of others.
3. Choosing sufficient set for PS, if one knows something about the model is a solved problem.
4. That any empirical test of the bias-reduction potential of PS, can only be generalized to cases where the causal relationships among covariates, observed and unobserved is the same.

CONCLUSIONS

Structural Causal Model (SCM), enriched with logic and graphs, provides:

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- Unifies the graphical, potential-outcome and structural equation approaches
- Provides friendly and formal solutions to century-old problems and confusions.