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Tot Prob: $\mathbf{P}(B) = \mathbf{P}(A_1 \cap B) + ... + \mathbf{P}(A_n \cap B) = \mathbf{P}(A_1)\mathbf{P}(B|A_1) + ... + \mathbf{P}(A_n)\mathbf{P}(B|A_n)$
Bayes: $\mathbf{P}(A|B) = \frac{\mathbf{P}(B|A)\mathbf{P}(A)}{\mathbf{P}(B)} = \frac{\mathbf{P}(B|A)\mathbf{P}(A)}{\mathbf{P}(B|A_i)\sum_{i=1}^{k}\mathbf{P}(A_i)}$

Counting	Order	No Order
Replacement	n^r	
No Replacement	$\frac{n!}{(n-r)!}$	$\frac{n!}{(n-r!)r!}$

$$Multinomial : f(x_1, ..., x_k; n, p_1, ..., p_k) = \mathbf{P}(X_1 = x_1, ..., X_k = x_k) = \frac{n!}{x_1! \cdot ... \cdot x_k!} p_1^{x_1} ... p_k^{x_k}$$

Name	*Definition*	$\mathbf{E}[X]$	$var(X)$	notes
Bernoulli:	$f_X(x) = \begin{cases} p, & \text{if } x = 1 \\ 1-p, & \text{if } x = 0 \end{cases}$	p	$p(1-p)$	
Binomial:	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$	np	$p(1-p)$	$\mathbf{P}(X \leq k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$
Geometric:	$p_X(k) = (1-p)^{k-1} p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\mathbf{P}(X > k) = (1-p^k)$ $\mathbf{P}(X < k) = 1 - (1-p^k)$ $\mathbf{P}(X = t) = \mathbf{P}(X = k+t-1 \mid X \geq k)$ $\mathbf{P}(X = t) = \mathbf{P}(X = k+t \mid X > k)$
Exponential:	$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\mathbf{P}(X \geq a) = e^{-\lambda a}$
Poisson	$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$	λ	λ	$\lambda = np$ for large n and small p

Conditional:	$p_{X\|A}(x) = \mathbf{P}(X = x\|A) = \frac{P(X=x \cap A)}{\mathbf{P}(A)}$	**Memoryless:**	$PR(X = k+t-1\|X \geq k) = \mathbf{P}(X = t)$

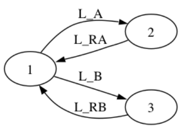
$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}$, $\sum_{i=0}^{\infty} c^i = \frac{1}{1-c}$, $\sum_{i=0}^{\infty} c^i = \frac{1}{1-c}$, $\sum_{i=0}^{n} ic^i = \frac{nc^{n+2}-(n+1)c^{n+1}+c}{(c-1)^2}$, $\sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}$ $|c| < 1$

$$\Lambda_i = \sum_j \Lambda_{j,i} \qquad \sum_j \Lambda_{j,i} = \sum_j \Lambda_{i,j} \qquad \Lambda_i = \lambda_i \pi_i \quad | \quad \Lambda_i = \frac{1}{T_i}$$

Flow Balance Equations - ($f_{j,i}$ = fraction of departures from s_j to s_i).

$\Lambda_i^O = \Lambda_i^I$	$\pi_i \sum_j \lambda_{i,j} = \sum_j \pi_j \lambda_{j,i}$	$\pi_i \lambda_i = \sum_j \pi_j \lambda_j f_{j,i}$	$\pi_i \frac{1}{T_i} = \sum_j \pi_j \frac{1}{T_j} f_{j,i}$	$\pi_i = \sum_j \pi_j f_{j,i}$(all T_i equal)

CTMC $Q = \left| \begin{matrix} -\lambda_A - \lambda_B & \lambda_A & \lambda_B \\ \lambda_{RA} & -\lambda_{RA} & 0 \\ \lambda_{RB} & 0 & -\lambda_{RB} \end{matrix} \right|$



$0 = (-\lambda_A - \lambda_B)\pi_1 + \lambda_{RA}\pi_2 + \lambda_{RB}\pi_3$
$0 = \lambda_A \pi_1 - \lambda_{RA}\pi_2$
$0 = \lambda_B \pi_1 - \lambda_{RB}\pi_3$
$\pi_1 + \pi_2 + \pi_3 = 1$

Little's Result - $\Lambda T = \bar{n}$ Λ = mean arrival rate to box, $\bar{n}$ =mean entities in box, T =mean time in box

M/M/1 - $\boldsymbol{\pi_{n-1}}\lambda = \pi_n \mu, \forall n \geq 1$ $\boldsymbol{\pi_0} = \frac{1}{\sum_{n=0}^{\infty} \rho^n} = 1 - \rho$ $\boldsymbol{\pi_n} = (1-\rho)\rho^n$ $\boldsymbol{\rho} = \frac{\lambda}{\mu}$ $\boldsymbol{\bar{n}} = \frac{\rho}{1-\rho}$ $\boldsymbol{T} = \frac{\bar{n}}{\lambda} = \frac{\frac{1}{\mu}}{1-\rho}$

$\mathbf{P}$(no wait) $= \pi_0$ $\mathbf{P}$(leave empty system) $= \frac{\pi_1 \mu}{\lambda} = \frac{\pi_0 \lambda}{\lambda} = \pi_0$

M/M/1 (state dependent) - $\boldsymbol{\pi_n}\mu_n = \pi_{n-1}\lambda_{n-1}, \forall n \geq 1$ $\boldsymbol{\pi_n} = \pi_0 \prod_{k=1}^{n} \frac{\lambda_{k-1}}{\mu_k}$ $\boldsymbol{\pi_0} = \frac{1}{1+\sum_{n=1}^{\infty}\prod_{k=1}^{n}\frac{\lambda_{k-1}}{\mu_k}}$

Finite Population $\boldsymbol{N}$ - Same except now $\sum_{n=1}^{N} \pi_n = 1$

Discouraged Arrivals $\boldsymbol{\lambda_n = \frac{\lambda}{n+1}}$, μ_n is constant: $\boldsymbol{\pi_n} = \pi_0 \frac{\lambda^n}{n!\mu^n} = \pi_0 \frac{\rho^n}{n!}$ $\boldsymbol{\pi_0} = \frac{1}{\sum_{n=0}^{\infty} \frac{\rho^n}{n!}} = \frac{1}{e^\rho} = e^{-\rho}$

Combined: $\boldsymbol{\pi_n} = \frac{\rho^n e^{-\rho}}{n!}$ (poisson) $\boldsymbol{\bar{n}} = \rho$ $\boldsymbol{\bar{\lambda}} = \mu(1-e^{-\rho}) = \mu(1-\pi_0)$ $\boldsymbol{T} = \frac{\bar{n}}{\lambda} = \frac{\rho}{\mu(1-e^{-\rho})}$ $\mathbf{lim}_{\boldsymbol{\rho \to 0}} \boldsymbol{T} = \frac{1}{\mu}$

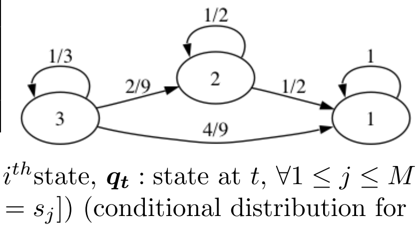
Infinite Servers: $\boldsymbol{\mu_n = n\mu}$: $\boldsymbol{\pi_n} = \pi_0 \frac{\rho^n}{n!}$ $\boldsymbol{T} = \frac{\bar{n}}{\lambda} = \frac{\rho}{\lambda} = \frac{1}{\mu}$

Finite Population: : **mean arrival rate:** $n\lambda$ **mean departure rate:** μ (except in state N when it's 0)

Finite Buffer Space: $\boldsymbol{\pi_n} = \frac{\lambda_{n-1}}{\mu}\pi_{n-1}$ mean conditional rate of loss: $\pi_N \lambda_N$ **Total arrival rate of customers:** $\sum_{n=0}^{N} \pi_n \lambda_n$ **Fraction lost customers**: $\frac{\pi_N \lambda_N}{\sum_{n=0}^{N} \pi_n \lambda_n}$. $\lambda_n = \lambda, \forall n \implies$ this reduces to π_N

Transient Analysis

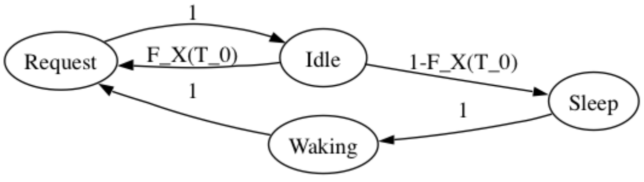
$\boldsymbol{P} = \left| \begin{matrix} \frac{3}{9} & \frac{2}{9} & \frac{4}{9} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{matrix} \right|$



$\prod^{(0)} = [100]$
$\prod^{(1)} = \prod^{(0)} * P = [\frac{3}{9} \frac{2}{9} \frac{4}{9}]$
$\prod^{(2)} = \prod^{(1)} * P = [\frac{1}{9} \frac{5}{27} \frac{19}{27}]$
$\mathbf{P}[N > 2] = 1 - P[N \leq 2] = 1 - \frac{19}{27} = \frac{8}{27}$

HMM - $\boldsymbol{N}$:#states, $\boldsymbol{s_i}$: i^{th}state, $\boldsymbol{q_t}$: state at t, $\forall 1 \leq j \leq M$, $\boldsymbol{v_k}$: distinct observations, $\boldsymbol{O_j}$: Observed value at j
$\boldsymbol{b_j(k)} = \mathbf{P}([O_t = v_k | q_t = s_j])$ (conditional distribution for observed value given state of the underlying MC),
$\boldsymbol{B} = \{b_j(k)\}$
$\boldsymbol{\alpha_t(O_t, q_t = s_i | M)} = \mathbf{P}[O_t, q_t = s_i | M]$ $\boldsymbol{\alpha_t(O_{t+1}, q_{t+1} = s_j | M)} = \sum_{i=1}^{N} \alpha_t(O_t, q_t = s_i | M) a_{i,j} b_j(O_{t+1})$
ie: $\alpha_1(o_1 = a, q_1 = q | M) = \pi[1] * b_1[a] = \pi[1] * P(a|1)$ |

Semi Markov 1



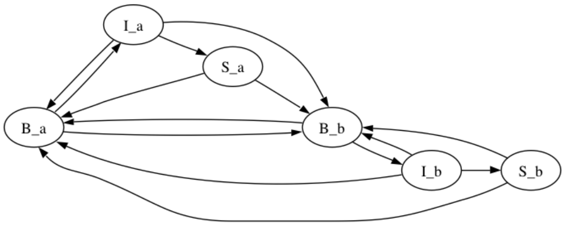
$P_{idle \rightarrow request} = \mathbf{P}(t \leq T_0) = F_X(T_0)$
$P_{idle \rightarrow sleep} = \mathbf{P}(t > T_0) = 1 - F_X(T_0)$
Mean Sojourn Time In:
$Request : E[R] = \int_0^\infty r f_R(r) dr$
$Idle : P[X \leq T_0] * E[X|X \leq T_0] + P[X > T_0] * T_0 =$
$\int_0^{T_0} x f_X(x) dx + (1 - F_X(T_0)) T_0$
$Sleep :$ (Remaining time in idle) $: f_Y(y) = \frac{f_X(y+T_0)}{1-F_X(T_0)}$
$E[Y] = \int_0^\infty y * f_Y(y) dy$

$\boldsymbol{V} = \boldsymbol{V}\boldsymbol{P}$ to solve for visit ratios to markov chain. Use V to solve for π, the fraction of time spent in each state.
$\pi_i = \frac{v_i T_i}{\sum_{i=1}^4 v_i T_i}$

Semi Markov 2

$$\text{time between arrivals} = f_X(t) = \begin{cases} \lambda_a e^{-\lambda_a t} \text{ with probability p} \\ \lambda_b e^{-\lambda_b t} \text{ with probability 1-p} \end{cases}$$

Service time is μ. After request done, if a server remains idle for T (a constant), then sleep.



Sojourn Times
$\mathbf{P}[(B,a)] = \frac{1}{\mu + \lambda_b}$
$\mathbf{P}[(B,b)] = \frac{1}{\mu + \lambda_a}$
$\mathbf{P}[(I,a)] = \int_0^T \lambda_a e^{-\lambda_a t} * t dt + T * (e^{-\lambda_a T}$
$\mathbf{P}[(I,b)] = \int_0^T \lambda_b e^{-\lambda_b t} * t dt + T * (e^{-\lambda_a T}$
$\mathbf{P}[(S,a)] = \frac{1}{\lambda_a}$
$\mathbf{P}[(S,b)] = \frac{1}{\lambda_b}$

Probabilities:

$B,a \rightarrow I,a \quad = p * [1 - e^{-\lambda_a T}]$
$B,b \rightarrow I,b \quad = \frac{\mu}{\lambda_a + \mu}$
$I,a \rightarrow S,a \quad = e^{-\lambda_a T}$
$I,b \rightarrow S,b \quad = e^{-\lambda_b T}$
$S,b \rightarrow B,a \quad = p$
$B,a \rightarrow B,b \quad = \frac{\lambda_B}{\lambda_B + \mu}$
$I,a \rightarrow B,a \quad = p * [1 - e^{-\lambda_a T)}]$
$I,b \rightarrow B,a \quad = (p) * [1 - e^{-\lambda_b T}]$
$S,a \rightarrow B,a \quad = p$
$S,b \rightarrow B,b \quad = 1 - p$
$B,b \rightarrow B,a \quad = \frac{\lambda_a}{\lambda_a + \mu}$
$I,a \rightarrow B,b \quad = (1-p) * [1 - e^{-\lambda_a T}]$
$I,b \rightarrow B,b \quad =$
$S,a \rightarrow B,b \quad = 1 - p$

PDF:

$\mathbf{P}(X \in B) = \int_B f_X(x) dx$ $\quad$ $\mathbf{P}(a \leq X \leq b) = \int_a^b f_X(x) dx$ $\quad$ $\mathbf{P}(-\infty < X < \infty) = \int_{-\infty}^\infty f_x(x) dx = 1$

$\mathbf{E}[X] = \int_{-\infty}^\infty x f_X(x) dx$ $\quad$ $\mathbf{E}[g(X)] = \int_{-\infty}^\infty g(x) f_X(x) dx$ $\quad$ $\mathbf{Var}(X) = \mathbf{E}[(X - \mathbf{E}[X])^2] = \int_{-\infty}^\infty (x = \mathbf{E}[X])^2 f_X(x) dx$

$0 \leq \mathbf{Var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$

$Y = aX + b \implies$ $\quad$ $\mathbf{E}[Y] = a\mathbf{E}[X] + b$ $\quad$ $\mathbf{Var}(Y) = a^2 \mathbf{Var}(X)$

$\mathbf{P}(X \in B|A) = \int_B f_{X|A}(x) dx$ $\quad$ $Pr(f_{X|A}(x)) = \begin{cases} \frac{f_X(x)}{\mathbf{P}(X \in A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ $\quad$ $\mathbf{P}(X \in B|X \in A) = \int_B f_{X|A}(x) dx$

$\mathbf{P}(X \in B|X \in A) = \int_B f_{X|A}(x) dx$ $\quad$ $\mathbf{E}[X|A] = \int_{-\infty}^\infty x f_{X|A}(x) dx$ $\quad$ $\mathbf{E}[g(X)|A] = \int_{-\infty}^\infty g(x) f_{X|A}(x) dx$

$disjoint\{A_1 ... A_n\} \implies f_x(x) = \sum_{i=1}^n \mathbf{P}(A_i) f_{X|A_i}(x)$ $\quad$ $\mathbf{E}[X] = \sum_{i=1}^n \mathbf{P}(A_i) \mathbf{E}[X|A_i]$ $\quad$ $\mathbf{E}[g(X)] = \sum_{i=1}^n \mathbf{P}(A_i) \mathbf{E}[g(X)|A_i]$

CDF:

$F_X(x) = \mathbf{P}(X \leq x), \forall x$ $\quad$ $x \leq y \implies F_X(x) \leq F_X(y)$ $\quad$ $F_X(k) = \sum_{i=-\infty}^k p_x(i)$

$p_X(k) = \mathbf{P}(X \leq k) - \mathbf{P}(X \leq k-1) = F_X(k) - F_x(k-1)$

$F_X(x) = \int_{-\infty}^x f_x(t) dt$ $\quad$ $f_X(x) = \frac{dF_X}{dx}(x)$
$f_X(x) = \int_{-\infty}^\infty f_{X,Y}(x,y) dy$ $\quad$ $f_Y(x) = \int_{-\infty}^\infty f_{X,Y}(x,y) dx.$
$\mathbf{E}[X] = \sum_x x \cdot p_X(x)$ $\quad$ $Y = g(x) \implies \mathbf{E}[Y] = \sum_x g(x) \cdot \mathbf{P}_X(x)$