A Graphical Criterion for the Identification of Causal Effects in Linear Models

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Introduction

- Find and estimate a model for data obtained from measurements on observable variables.

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- Structural Equation Models (SEM)

  Set of linear equations.

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  Set of linear equations.
- Natural causal interpretation

History

- Method of Path Coefficients (Wright, 34)
- Economics (Fisher, 66; Bowden & Turkington, 84)
- Social Sciences (Duncan, 75)
- Graphical Methods (Spirtes etal,93; Pearl,95; McDonald,97)

Anatomy and Causal Interpretation of a linear equation

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- β quantifies the Causal-Effect of X on Y.
- e represents unobserved background factors.

Linear Models

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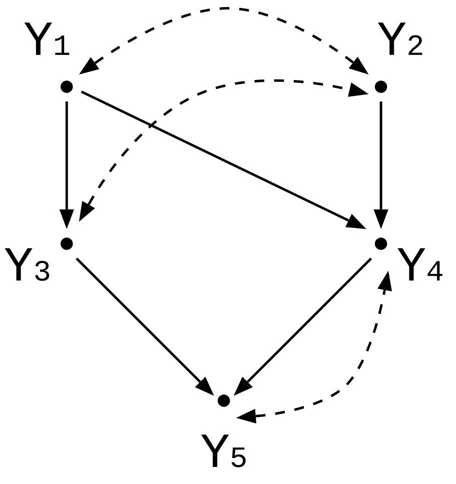

structure

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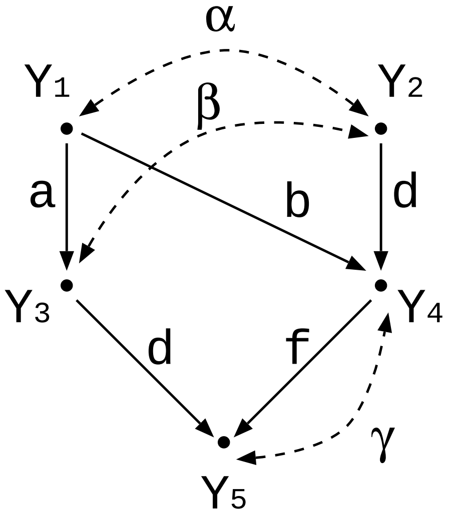
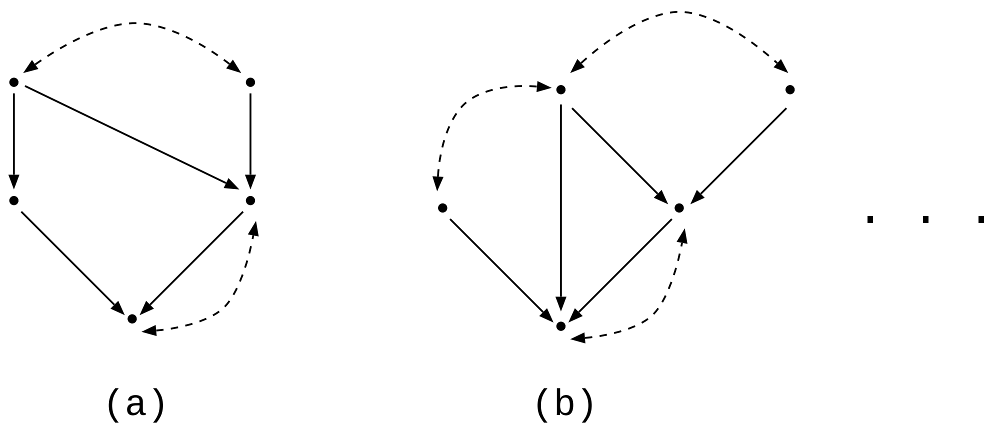


structure + parameters

Problems

- Find and Test the model structure



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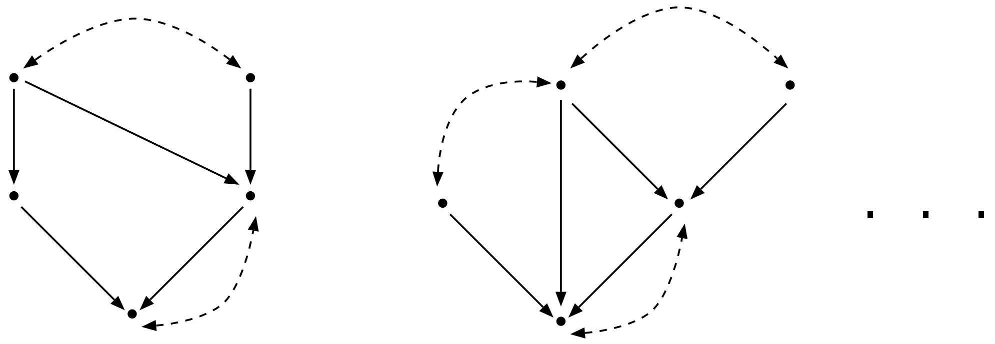

(a) (b)

- The Identification Problem
  - Computing the parameters

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- Given:
  - Covariance matrix Σ over $\{Y_1, \ldots, Y_n\}$
  - Structure of the model

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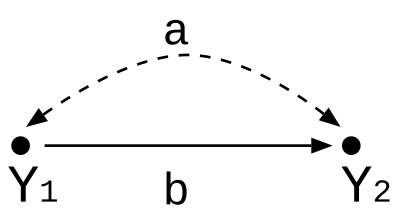
- Existence and Uniqueness
  - No parametrization makes the model generate Σ.
  - There may be more than one solution.



Traditional Approaches

- Based on algebraic manipulations of the equations defining the model

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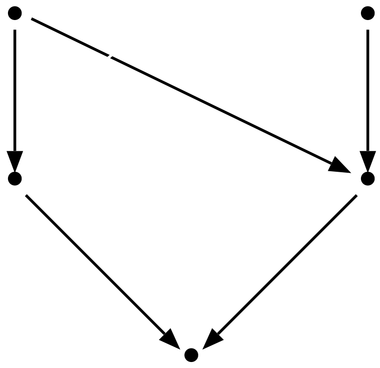
- Instrumental Variables [Bowden & Turkington, 84]

Graphical Criteria

- McDonald's Regressional Hierarchy (McDonald, 1997)



(a) No bidirected edges.

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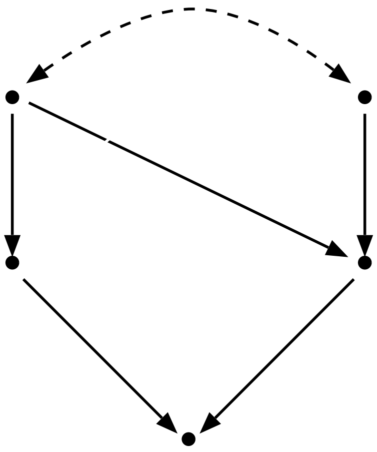

(b) Bidirected edges only between root variables.

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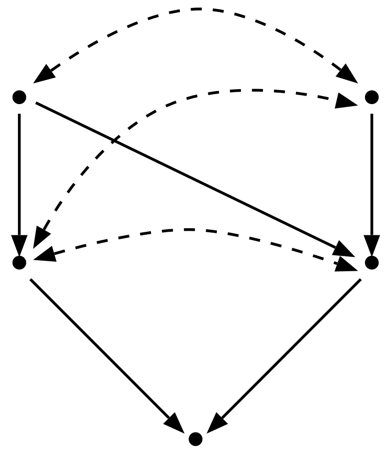
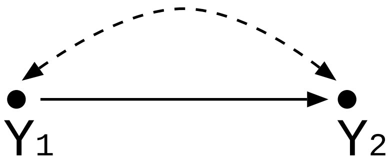


(c) Bidirected edges only between not causally related variables.

Graphical Criteria (contd)

- Bow-pattern



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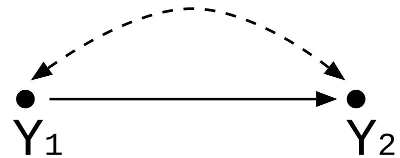
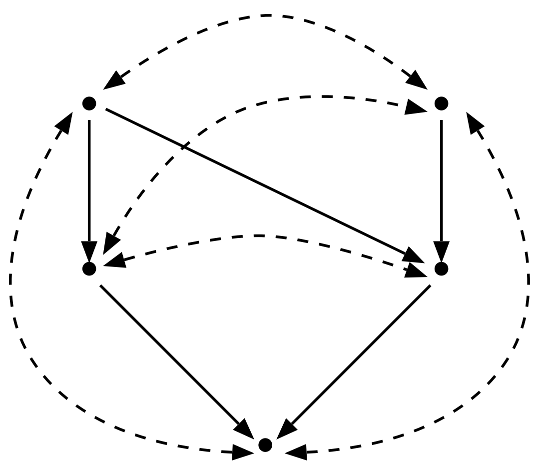


- Bow-free models (Brito & Pearl, 2002)



Goal

- Find graphical conditions on the structure of a general model that guarantee the identification of all parameters.

Strategy

- Fix a variable Y, and assume that the parameters of edges at depth lower than Y are all identified.

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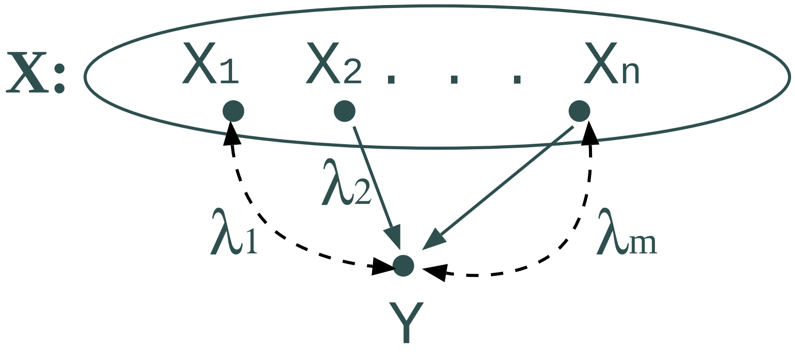

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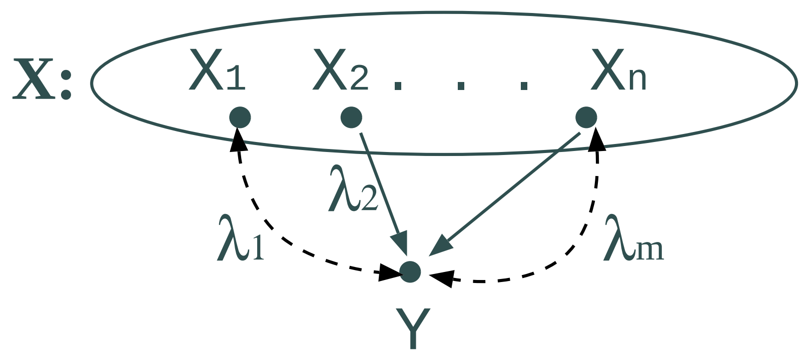

- Establish graphical conditions on the causal graph such that the parameters $\lambda_1, \ldots, \lambda_m$ are identified.

Wright's Method of Path Coefficients

- Correlation coefficient of X and Y can be expressed as:

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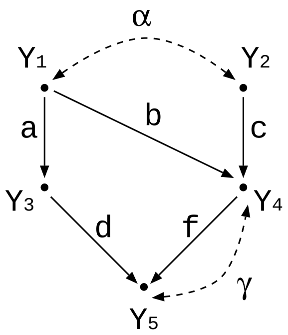
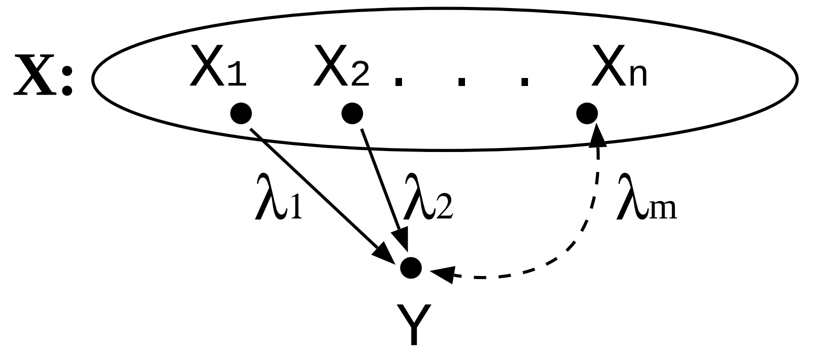
- Example:



$\rho_{12} = \alpha$
$\rho_{14} = b + \alpha c$
$\rho_{34} = ab + a\alpha c$
$\rho_{15} = ad + bf + \alpha cf$
$\rho_{45} = f + \gamma + c\alpha ad + bad$

Earlier Result (Brito & Pearl, 2002)



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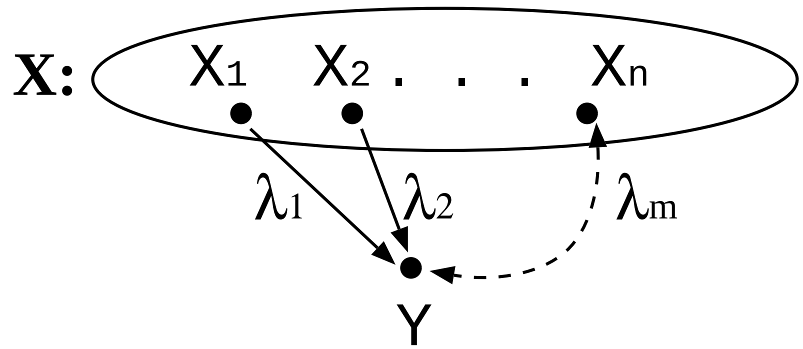

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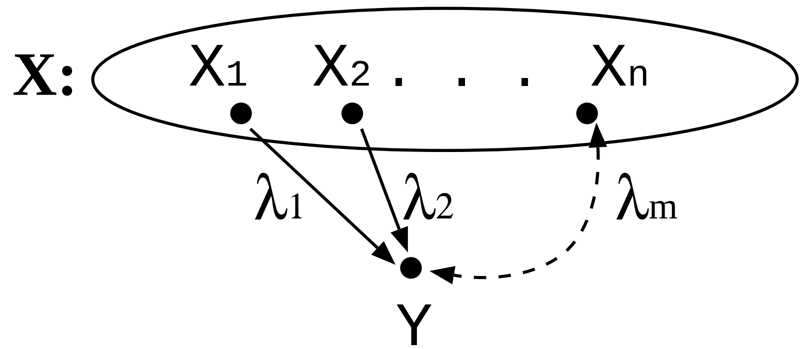

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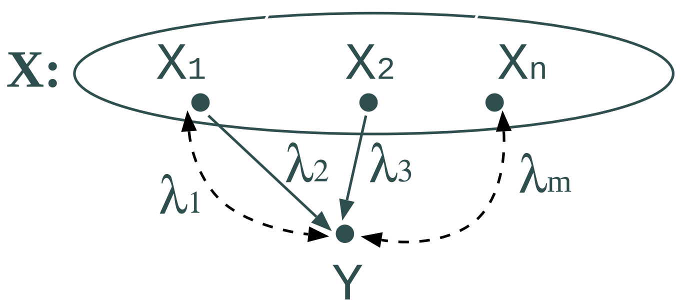
COROLLARY: Bow-free models are completely identifiable.

Auxiliary Variables



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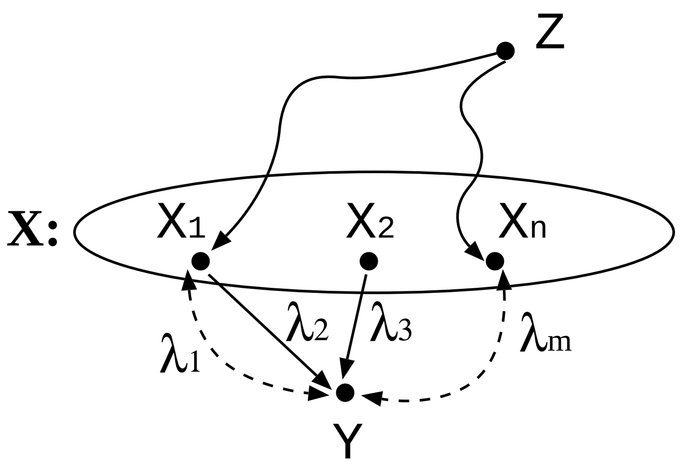

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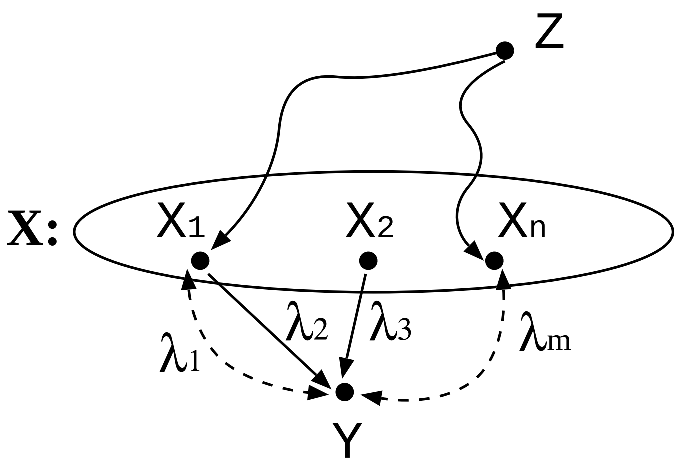

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DEFINITION: Variable Z is an Auxiliary Variable iff Wright's equation for (Z, Y) is linearly indenpendent from Φ.

Graphical Criterion:

A Chain (or, auxiliary chain) connecting Z and Y is a sequence of paths between $Z, X_1, \ldots, X_k, Y$ such that

(i) there is an unblocked path p_j between X_j and X_{j+1} pointing to both of them;

(ii) there is an unblocked path p_k between X_k and Z pointing to X_k;

(iii) If some $X_l \in \mathbf{X}$ is an intermediate variable of path p_j, then X_l is connected to Y by a bidirected edge and subpath $p_j[X_l \sim X_{j+1}]$ points to X_l.

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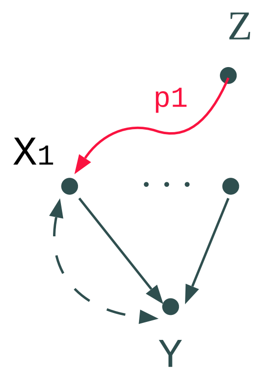
THEOREM: Variable Z is an Auxiliary Variable if and only if there is a chain connecting Z and Y.

Intuition and Examples



simple case

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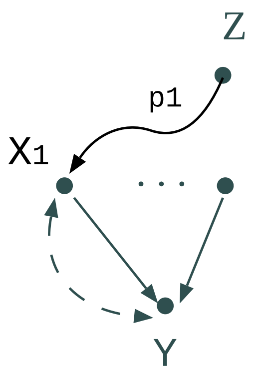

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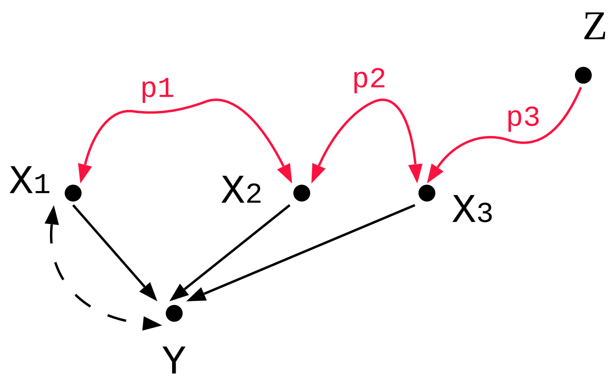
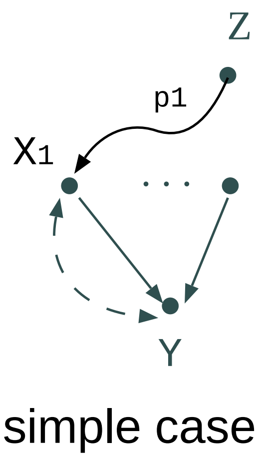
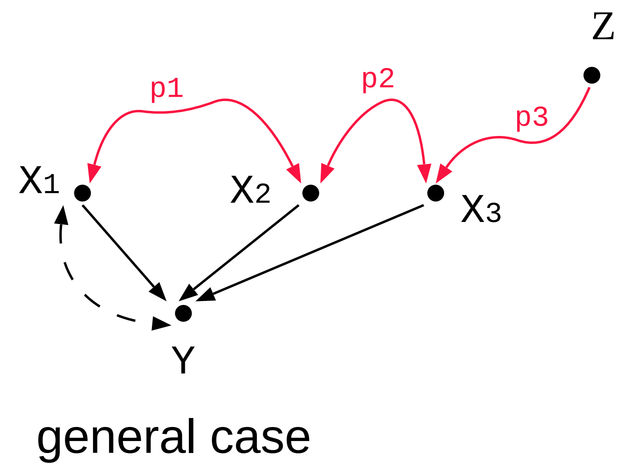


general case

Intuition and Examples





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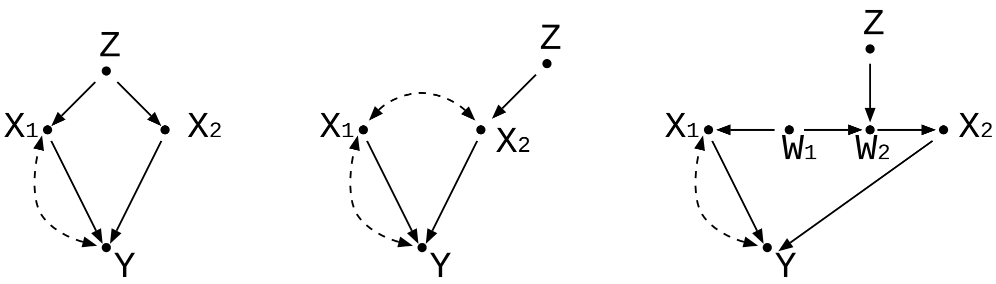

(a) (b) (c)

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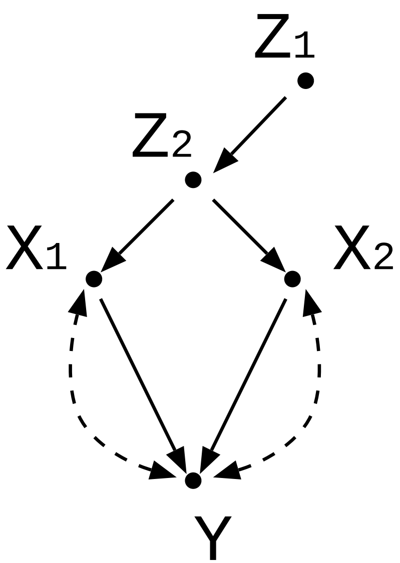

GAV Criterion:

A set $\{Z_1, \ldots, Z_k\}$ of Auxiliary variables satisfies the GAV criterion if:

(i) C_i is an auxiliary chain between Z_i and X_i;

(ii) For any two auxiliary chains C_i and C_j:

(a) C_i and C_j have no common variables other than Z_i or Z_j;

(b) If Z_i appears in path p_l of C_j, then both the subpath $p_i[Z_i \sim X_{l+1}]$ and the last path of C_i point to Z_i.

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THEOREM: The GAV criterion is a sufficient condition for a set **Z** of Auxiliary variables to allow identification.

Graphical Criterion for Identification

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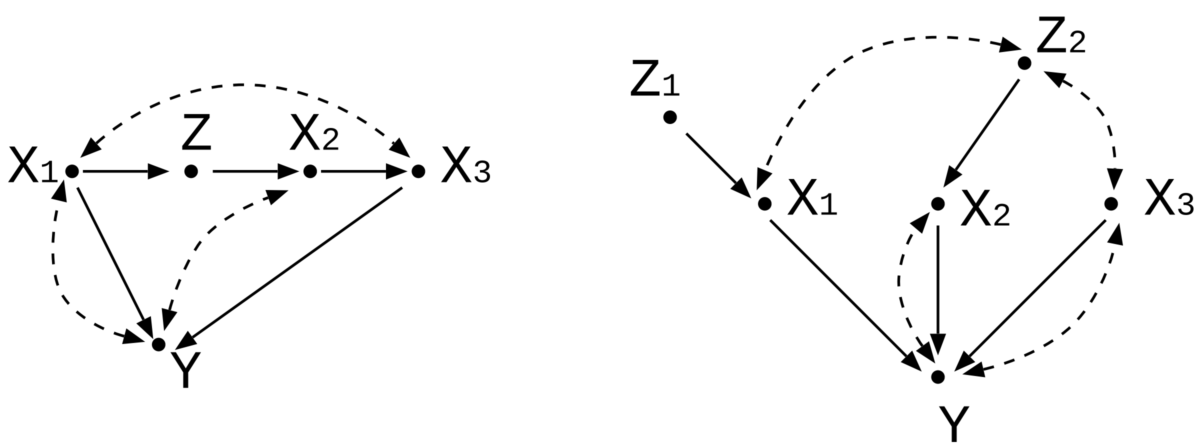

(a) (b)

Graphical Criterion for Identification

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Drawback: The criterion is not simple to be easily verified by inspection of the graph.

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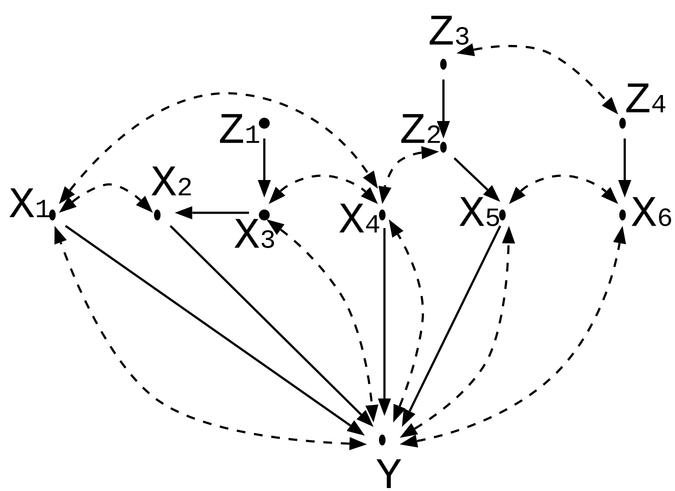
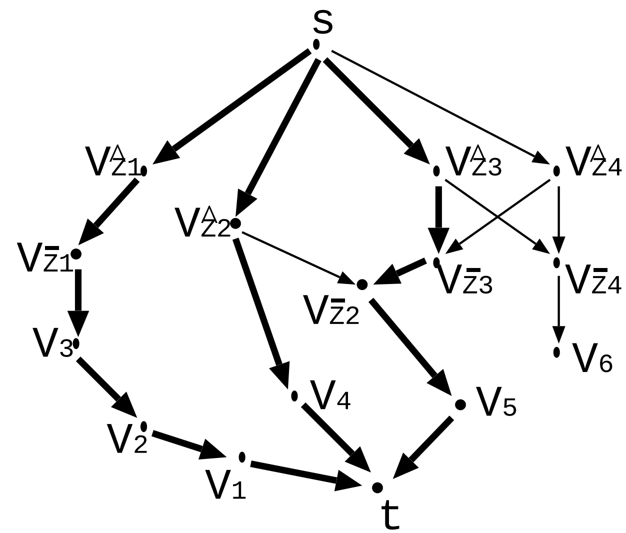
- **THEOREM:** The algorithm is Sound and Complete.

Example


Z3
Z4
Z1
Z2
X1
X2
X3
X4
X5
X6
Y


s
V△Z1
V△Z2
V△Z3
V△Z4
V=Z1
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- A new Graphical criterion for Identification.
- The GAV criterion does not make direct use of conditional independence.
- Algorithm to find appropriate Auxiliary variables.