THE INTERSECTION OF TWO HALFSPACES HAS HIGH THRESHOLD DEGREE*

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Abstract. The *threshold degree* of a Boolean function $f\colon \{0,1\}^n \to \{-1,+1\}$ is the least degree of a real polynomial p such that $f(x) \equiv \operatorname{sgn} p(x)$. We construct two halfspaces on $\{0,1\}^n$ whose intersection has threshold degree $\Theta(\sqrt{n})$, an exponential improvement on previous lower bounds. This solves an open problem due to Klivans (2002) and rules out the use of perceptron-based techniques for PAC learning the intersection of two halfspaces, a central unresolved challenge in computational learning. We also prove that the intersection of two majority functions has threshold degree $\Omega(\log n)$, which is tight and settles a conjecture of O'Donnell and Servedio (2003).

Our proof consists of two parts. First, we show that for any nonconstant Boolean functions f and g, the intersection $f(x) \wedge g(y)$ has threshold degree $O(d)$ if and only if $\|f-F\|_\infty + \|g-G\|_\infty < 1$ for some rational functions F, G of degree $O(d)$. Second, we determine the least degree required for approximating a halfspace and a majority function to any given accuracy by rational functions.

Our technique further allows us to obtain direct sum theorems for polynomial representations of composed Boolean functions. In particular, we give an improved lower bound on the approximate degree of the AND-OR tree.

Key words. intersections of halfspaces, polynomial representations of Boolean functions, rational approximation, direct sum theorems, PAC learning

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1. Introduction.

Representations of Boolean functions by real polynomials play an important role in theoretical computer science, with applications ranging from complexity theory to quantum computing and learning theory. Surveys in [8, 39, 14, 43] offer a glimpse into the diversity of these results and techniques. We study one such representation scheme known as *sign-representation*. Specifically, fix a Boolean function $f\colon X \to \{-1,+1\}$ for some finite set $X \subset \mathbb{R}^n$, such as the hypercube $X = \{-1,+1\}^n$. The *threshold degree* of f, denoted $\deg_\pm(f)$, is the least degree of a polynomial $p(x_1,\ldots,x_n)$ such that

$$f(x) = \operatorname{sgn} p(x)$$

for each $x \in X$. In other words, the threshold degree of f is the least degree of a real polynomial that represents f in sign.

The formal study of this complexity measure and of sign-representations in general began in 1969 with the seminal work of Minsky and Papert [31], who examined the threshold degree of several common functions. Since then, sign-representations have found a variety of applications. Paturi and Saks [36] and later Siu et al. [46] used Boolean functions with high threshold degree to obtain size-depth trade-offs for threshold circuits. The well-known result, due to Beigel et al. [10], that PP is closed under intersection is also naturally interpreted in terms of threshold degree. In another development, Aspnes et al. [7] used the notion of threshold degree and its relaxations to obtain oracle separations for PP and to give an insightful new proof of classical lower bounds for AC^0. Krause and Pudlák [27, 28] used random restrictions

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to show that the threshold degree gives lower bounds on the *weight* and *density* of perceptrons and their generalizations, which are well-studied computational models.

Learning theory is another area in which the threshold degree of Boolean functions is of interest. Specifically, functions with low threshold degree can be efficiently PAC learned under arbitrary distributions, via linear programming. The current fastest algorithm for PAC learning polynomial-size DNF formulas, due to Klivans and Servedio [23], is an illustrative example: it is based precisely on an upper bound on the threshold degree of this concept class.

The threshold degree has recently become a tool in communication complexity. The starting point in this line of work is the Degree/Discrepancy Theorem [42, 41], which states that any Boolean function with high threshold degree induces a communication problem with low *discrepancy* and thus high communication complexity in almost all models. This result was used in [42] to show the optimality of Allender's simulation of AC^0 by majority circuits [4], thus solving an open problem of Krause and Pudlák [27]. Known lower bounds on the threshold degree have played an important role in recent progress [44, 37] on *unbounded-error* communication complexity, which is more powerful than the models above.

Despite these applications, analyzing the threshold degree has remained a difficult task, and Minsky and Papert's *symmetrization* technique from 1969 has been essentially the only method available. Unfortunately, symmetrization only applies to symmetric Boolean functions and certain derivations thereof. In a recent tutorial, Aaronson [2] re-posed the challenge of developing new analytic techniques for multivariate real polynomials that represent Boolean functions. We make progress on this challenge in the context of sign-representation, contributing a number of direct sum theorems for the threshold degree. As an application, we construct two halfspaces on $\{0,1\}^n$ whose intersection has threshold degree $\Omega(\sqrt{n})$, which solves an open problem due to Klivans [21] and rules out the use of perceptron-based techniques for PAC learning the intersection of even two halfspaces. We give a detailed description of our results in Sections 1.1–1.3, followed by a discussion of our techniques in Section 1.4.

1.1. Results for general compositions. Very broadly, a *direct sum theorem* states that solving k independent instances of a problem requires $\Omega(k)$ times the resources for a single instance. In the context of sign-representation or uniform approximation, the relevant resource is polynomial degree. Our first result is a direct sum-type theorem for the threshold degree of composed functions.

THEOREM 1.1 (Threshold degree). *Consider functions* $f\colon X \to \{-1,+1\}$ *and* $F\colon \{-1,+1\}^k \to \{-1,+1\}$, *where* $X \subset \mathbb{R}^n$ *is a finite set. Then*

$$\deg_\pm(F(f,\dots,f)) \geqslant \deg_\pm(F)\deg_\pm(f).$$

Theorem 1.1 gives the best possible lower bound that depends on $\deg_\pm(F)$ and $\deg_\pm(f)$ alone. In particular, the bound is tight whenever $F = \text{PARITY}$ or $f = \text{PARITY}$. To our knowledge, the only previous direct sum theorem for the threshold degree was the XOR lemma in [34], which states that the XOR of k copies of a given function $f\colon X \to \{-1,+1\}$ has threshold degree $k\deg_\pm(f)$.

We generalize Theorem 1.1 to the notion of ϵ-*approximate degree* $\deg_\epsilon(F)$, which is the least degree of a real polynomial p with $\|F - p\|_\infty \leqslant \epsilon$. This notion plays a fundamental role in complexity theory, learning theory, and quantum computing and was also re-posed as an analytic challenge in Aaronson's tutorial [2]. We have:

THEOREM 1.2 (Approximate degree). *Fix functions* $f\colon X \to \{-1,+1\}$ *and*

$F\colon \{-1,+1\}^k \to \{-1,+1\}$, *where* $X \subset \mathbb{R}^n$ *is a finite set. Then for* $0 < \epsilon < 1$,

$$\deg_\epsilon(F(f,\ldots,f)) \geqslant \deg_\epsilon(F)\deg_\pm(f).$$

Again, Theorem 1.2 gives the best lower bound that depends on $\deg_\epsilon(F)$ and $\deg_\pm(f)$ alone. For example, the stated bound is tight for any function F when $f = \text{PARITY}$. In Section 3.1, we prove various other results involving bounded-error and small-bias approximation, as well as compositions of the form $F(f_1,\ldots,f_k)$ where $f_1,\ldots,f_k$ may all be distinct.

We use Theorem 1.2 to obtain an improved lower bound on the approximate degree of the AND-OR tree, given by

$$f(x) = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{n} x_{ij}. \tag{1.1}$$

Prior to this work, the best lower bound was $\Omega(n^{0.66\cdots})$, due to Ambainis [5]. Preceding it were lower bounds of $\Omega(\sqrt{n})$ due to Nisan and Szegedy [33] and $\Omega(\sqrt{n\log n})$ due to Shi [45]. We improve the standing lower bound to $\Omega(n^{0.75})$, the best upper bound being $O(n)$ due to Høyer et al. [18].

Theorem 1.3 (AND-OR Tree). *Define* $f\colon \{-1,+1\}^{n^2} \to \{-1,+1\}$ *by* (1.1). *Then*

$$\deg_{1/3}(f) = \Omega(n^{0.75}).$$

We find the proof of Theorem 1.3 in this paper simpler than the previous lower bound [5], which was based on the collision and element distinctness problems.

1.2. Results for specific compositions. While Theorems 1.1 and 1.2 give the best lower bounds that depend on $\deg_\pm(F)$, $\deg_\pm(f)$, and $\deg_\epsilon(F)$ alone, stronger lower bounds can be derived in some cases by exploiting additional structure of F and f. Consider the special but illustrative case of the conjunction of two functions. In other words, we are given functions $f\colon X \to \{-1,+1\}$ and $g\colon Y \to \{-1,+1\}$ for some finite sets $X, Y \subset \mathbb{R}^n$ and would like to determine the threshold degree of their conjunction, $(f \wedge g)(x,y) = f(x) \wedge g(y)$. A simple and elegant method for sign-representing $f \wedge g$, due to Beigel et al. [10], is to use rational approximation. Specifically, let $p_1(x)/q_1(x)$ and $p_2(y)/q_2(y)$ be rational functions of degree d that approximate f and g, respectively, in the following sense:

$$\max_{x\in X}\left|f(x) - \frac{p_1(x)}{q_1(x)}\right| + \max_{y\in Y}\left|g(y) - \frac{p_2(y)}{q_2(y)}\right| < 1. \tag{1.2}$$

Letting -1 and $+1$ correspond to "true" and "false," respectively, we obtain:

$$f(x) \wedge g(y) \equiv \operatorname{sgn}\{1 + f(x) + g(y)\} \equiv \operatorname{sgn}\left\{1 + \frac{p_1(x)}{q_1(x)} + \frac{p_2(y)}{q_2(y)}\right\}. \tag{1.3}$$

Multiplying the last expression in braces by the positive quantity $q_1(x)^2 q_2(y)^2$ gives

$$f(x) \wedge g(y) \equiv \operatorname{sgn}\left\{q_1(x)^2q_2(y)^2 + p_1(x)q_1(x)q_2(y)^2 + p_2(y)q_1(x)^2q_2(y)\right\},$$

whence $\deg_\pm(f \wedge g) \leqslant 4d$. In summary, if f and g can be approximated as in (1.2) by rational functions of degree at most d, then the conjunction $f \wedge g$ has threshold degree at most $4d$.

It is natural to ask whether there exists a better construction. After all, given a sign-representing polynomial $p(x,y)$ for $f(x) \wedge g(y)$, there is no reason to expect that p arises from the sum of two independent rational functions as in (1.3). Indeed, x and y can be tightly coupled inside $p(x,y)$ and can interact in complicated ways. Our next result is that, surprisingly, no such interactions can beat the simple construction above. In other words, the sign-representation based on rational functions always achieves the optimal degree, up to a small constant factor.

THEOREM 1.4 (Conjunctions of functions). *Let $f\colon X \to \{-1,+1\}$ and $g\colon Y \to \{-1,+1\}$ be given functions, where $X, Y \subset \mathbb{R}^n$ are arbitrary finite sets. Assume that f and g are not identically false. Let $d = \deg_\pm(f \wedge g)$. Then there exist degree-$4d$ rational functions*

$$\frac{p_1(x)}{q_1(x)}, \quad \frac{p_2(y)}{q_2(y)}$$

that satisfy (1.2).

Via repeated applications of Theorem 1.4, we obtain analogous results for conjunctions $f_1 \wedge f_2 \wedge \cdots \wedge f_k$ for any Boolean functions $f_1, f_2, \ldots, f_k$ and any k. Our results further extend to compositions $F(f_1, \ldots, f_k)$ for various F other than $F = \text{AND}$, such as halfspaces and read-once AND/OR/NOT formulas. We defer a more detailed description of these extensions to Section 3.4, limiting this overview to the following representative special case.

THEOREM 1.5 (Extension to multiple functions). *Let $f_1, f_2, \ldots, f_k$ be non-constant Boolean functions on finite sets $X_1, X_2, \ldots, X_k \subset \mathbb{R}^n$, respectively. Let $F\colon \{-1,+1\}^k \to \{-1,+1\}$ be a halfspace or a read-once AND/OR/NOT formula. Assume that F depends on all of its k inputs and that the composition $F(f_1, f_2, \ldots, f_k)$ has threshold degree d. Then there is a degree-D rational function p_i/q_i on X_i, $i = 1, 2, \ldots, k$, such that*

$$\sum_{i=1}^{k} \max_{x_i \in X_i} \left| f_i(x_i) - \frac{p_i(x_i)}{q_i(x_i)} \right| < 1,$$

where $D = 8d \log 2k$.

Theorem 1.5 is close to optimal. For example, when $F = \text{AND}$, the upper bound on D is tight up to a factor of $\Theta(k \log k)$; for all F in the statement of the theorem, it is tight up to a polynomial in k. See Remark 3.21 for details.

Prior to this paper, it was a challenge to analyze the threshold degree even for compositions of the form $f \wedge g$. Indeed, we are only aware of the work in [31, 34], where the threshold degree of $f \wedge g$ was studied for the special case $f = g = \text{MAJORITY}$. The main difficulty in those previous works was analyzing the unintuitive interactions between f and g. Our results remove this difficulty, even in the general setting of compositions $F(f_1, f_2, \ldots, f_k)$ for arbitrary $f_1, f_2, \ldots, f_k$ and various combining functions F. Specifically, Theorems 1.4 and 1.5 make it possible to study the base functions $f_1, f_2, \ldots, f_k$ individually, in isolation. Once their rational approximability is understood, one immediately obtains lower bounds on the threshold degree of $F(f_1, f_2, \ldots, f_k)$.

1.3. Results for intersections of two halfspaces. As an application of our direct sum theorems in Section 1.2, we obtain the first strong lower bounds on the threshold degree of intersections of halfspaces, i.e., intersections of functions of the form $f(x) = \operatorname{sgn}(\sum \alpha_i x_i - \theta)$ for some reals $\alpha_1, \ldots, \alpha_n, \theta$. In light of Theorem 1.4, this task amounts to proving that rational functions of low degree cannot approximate a given halfspace. We do so in the following theorem, where the notation $\operatorname{rdeg}_\epsilon(f)$ stands for the least degree of a rational function A with $\|f - A\|_\infty \leqslant \epsilon$.

THEOREM 1.6 (Approximation of a halfspace). *Let* $f\colon \{-1,+1\}^{n^2} \to \{-1,+1\}$ *be given by*

$$f(x) = \operatorname{sgn}\left(1 + \sum_{i=1}^{n}\sum_{j=1}^{n} 2^i x_{ij}\right). \tag{1.4}$$

Then for $1/3 < \epsilon < 1$,

$$\operatorname{rdeg}_\epsilon(f) = \Theta\left(1 + \frac{n}{\log\{1/(1-\epsilon)\}}\right).$$

Furthermore, for all $\epsilon > 0$,

$$\operatorname{rdeg}_\epsilon(f) \leqslant 64n\lceil \log_2 n\rceil + 1.$$

The function (1.4) is known as the *canonical halfspace* [16]. Theorem 1.6 shows that a rational function of degree $\Theta(n)$ is necessary and sufficient for approximating the canonical halfspace within 1/3. The upper bound in this theorem follows readily from classical work by Newman [32], and it is the lower bound that has required of us technical novelty and effort. The best previous degree lower bound for constant-error approximation for any halfspace was $\Omega(\log n/\log\log n)$, obtained implicitly in [34]. We complement Theorem 1.6 with a solution for another common halfspace, the majority function.

THEOREM 1.7 (Approximation of majority). *Let* $\mathrm{MAJ}_n\colon \{-1,+1\}^n \to \{-1,+1\}$ *denote the majority function. Then*

$$\operatorname{rdeg}_\epsilon(\mathrm{MAJ}_n) = \begin{cases} \Theta\left(\log\left\{\dfrac{2n}{\log(1/\epsilon)}\right\}\cdot\log\dfrac{1}{\epsilon}\right), & 2^{-n} < \epsilon < 1/3, \\[2ex] \Theta\left(1 + \dfrac{\log n}{\log\{1/(1-\epsilon)\}}\right), & 1/3 \leqslant \epsilon < 1. \end{cases}$$

Again, the upper bound in Theorem 1.7 is relatively straightforward. Indeed, an upper bound of $O(\log\{1/\epsilon\}\log n)$ for $0 < \epsilon < 1/3$ was known and used in the complexity literature long before our work [36, 46, 10, 22], and we only somewhat tighten that upper bound and extend it to all ϵ. Our primary contribution in Theorem 1.7, then, is a matching *lower* bound on the degree, which requires new ideas. The closest previous line of research concerns *continuous* approximation of the sign function on $[-1,-\epsilon]\cup[\epsilon,1]$, which unfortunately gives no insight into the discrete case. For example, the lower bound derived by Newman [32] in the continuous setting is based on the integration of relevant rational functions with respect to a suitable weight function, which has no meaningful discrete analogue. We discuss our solution in greater detail at the end of the Introduction.

Our first application of these lower bounds for rational approximation is to construct an intersection of two halfspaces with high threshold degree. In what follows, the symbol $f \wedge f$ denotes the conjunction of two independent copies of a given Boolean function f, on disjoint sets of variables.

THEOREM 1.8 (Intersection of two halfspaces). *Let* $f\colon \{-1,+1\}^{n^2} \to \{-1,+1\}$ *be given by* (1.4). *Then*

$$\deg_{\pm}(f \wedge f) = \Omega(n).$$

For the specific halfspace f in Theorem 1.8, the lower bound is tight and matches the construction by Beigel et al. [10]. The best previous lower bound on the threshold degree of the intersection of two halfspaces was $\Omega(\log n/\log\log n)$, due to O'Donnell and Servedio [34], preceded in turn by an $\omega(1)$ lower bound due to Minsky and Papert [31].

Theorem 1.8 solves an open problem in computational learning theory, due to Klivans [21]. In more detail, recall that Boolean functions with low threshold degree can be efficiently PAC learned under arbitrary distributions, by expressing an unknown function as a perceptron with unknown weights and solving the associated linear program [23, 22]. A central challenge in the area is PAC learning the intersection of two halfspaces under arbitrary distributions, which remains unresolved despite much effort and solutions to some restrictions of the problem, e.g., [29, 47, 6, 22, 24]. Prior to this paper, it was unknown whether intersections of two halfspaces on the hypercube $\{0,1\}^n$ are amenable to learning via perceptron-based techniques. Specifically, Klivans [21, Sec. 7] asked for a lower bound of $\Omega(\log n)$ or better on the threshold degree of the intersection of two halfspaces on $\{0,1\}^n$. We solve this problem with a lower bound of $\Omega(\sqrt{n})$, thereby ruling out the use of perceptron-based techniques for learning the intersection of two halfspaces in subexponential time. To our knowledge, Theorem 1.8 is the first unconditional, structural lower bound for PAC learning the intersection of two halfspaces; all previous hardness results for the problem were based on complexity-theoretic assumptions [11, 3, 26, 20]. We complement Theorem 1.8 as follows.

THEOREM 1.9 (Mixed intersection). *Let* $f\colon \{-1,+1\}^{n^2} \to \{-1,+1\}$ *be given by* (1.4). *Let* $g\colon \{-1,+1\}^{\lceil\sqrt{n}\rceil} \to \{-1,+1\}$ *be the majority function. Then*

$$\deg_{\pm}(f \wedge g) = \Theta(\sqrt{n}).$$

In words, even if one of the halfspaces in Theorem 1.8 is replaced by a majority function, the threshold degree will remain high, resulting in a hard learning problem. Finally, we have:

THEOREM 1.10 (Intersection of two majorities). *Consider the majority function* $\mathrm{MAJ}_n\colon \{-1,+1\}^n \to \{-1,+1\}$. *Then*

$$\deg_{\pm}(\mathrm{MAJ}_n \wedge \mathrm{MAJ}_n) = \Omega(\log n).$$

Theorem 1.10 is tight, matching the construction of Beigel et al. [10]. It proves a conjecture of O'Donnell and Servedio [34], who gave a lower bound of $\Omega(\log n/\log\log n)$ with entirely different techniques and conjectured that the true answer was $\Omega(\log n)$. Theorems 1.8–1.10 are of course also valid for disjunctions rather than conjunctions.

Furthermore, Theorems 1.8 and 1.10 remain tight with respect to conjunctions of any constant number of functions.

Finally, we believe that the lower bounds for rational approximation in Theorems 1.6 and 1.7 are of independent interest. Rational functions are classical objects with various applications in theoretical computer science [10, 36, 46, 22, 1], and yet our ability to prove strong lower bounds for the rational approximation of Boolean functions has seen little progress since the seminal work in 1964 by Newman [32]. To illustrate some of the counterintuitive phenomena involved in rational approximation, consider the familiar function $\mathrm{OR}_n\colon \{0,1\}^n \to \{-1,+1\}$, given by $\mathrm{OR}_n(x) = 1 \Leftrightarrow x = 0$. A well-known result of Nisan and Szegedy [33] states that $\deg_{1/3}(\mathrm{OR}_n) = \Theta(\sqrt{n})$, meaning that a polynomial of degree $\Theta(\sqrt{n})$ is required for approximation within 1/3. At the same time, we claim that $\mathrm{rdeg}_\epsilon(\mathrm{OR}_n) = 1$ for all $0 < \epsilon < 1$. Indeed, let

$$A_M(x) = \frac{1 - M\sum x_i}{1 + M\sum x_i}.$$

Then $\|\mathrm{OR}_n - A_M\|_\infty \to 0$ as $M \to \infty$. We hope that Theorems 1.6 and 1.7 in this paper will encourage further progress on the rational approximation of Boolean functions.

1.4. Our techniques. We use one set of techniques to obtain the direct sum theorems for the threshold degree (Sections 1.1 and 1.2) and another, unrelated set of techniques to analyze the rational approximation of halfspaces (Section 1.3). We will give a separate overview of the technical development in each case.

Direct sum theorems. In symmetrization, one takes an assumed multivariate polynomial p that sign-represents a given symmetric function and converts p into a univariate polynomial, which is amenable to direct analysis. No such approach works for the function compositions of this paper, whose sign-representing polynomials can have complicated structure and will not simplify in a meaningful way.

Instead, our results are based on a detailed study of the *linear programming dual* of the sign-representation problems at hand. The challenge is to bring out, through the dual representation, analytic properties that will obey a direct sum theorem. Depending on the context (Theorem 1.1, 1.2, or 1.4), the property in question can be nonnegativity, correlation, orthogonality, certain quotient structure, or a combination of several of these. A strength of this approach is that it works with the sign-representation problem itself (over which we have considerable control) rather than an assumed sign-representing polynomial (whose structure we can no longer control in a meaningful way). To our knowledge, the idea of using dual representations to prove direct sum theorems for polynomials originates in the work of O'Donnell and Servedio [34], who showed that the XOR of k copies of a Boolean function f has threshold degree $k\deg_\pm(f)$.

As an illustration, we briefly describe the proof of Theorem 1.4. The dual object with which we work there is a certain problem of finding, in the column cones of two given matrices, two vectors whose corresponding entries have comparable magnitude. By an analytic argument, we are able to prove that this intermediate problem has the sought direct-sum property, giving the needed link between sign-representation and rational approximation. Thus, by working with the dual, we *implicitly* decompose any sign-representation $p(x,y)$ of the function $f(x) \wedge g(y)$ into individual rational approximants for f and g, regardless of how tightly the x and y parts are coupled inside p.

Rational approximation. The proof of Theorem 1.6 is built around two key ideas. The first is a new technique for placing lower bounds on the degree of a given polynomial $p \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ with prescribed approximate behavior, whereby one constructs a degree-nonincreasing linear map $M\colon \mathbb{R}[x_1, x_2, \ldots, x_n] \to \mathbb{R}[x]$ and argues that Mp has high degree. This technique is key to proving Theorem 1.6, which is not amenable to standard techniques such as symmetrization. As applied in this work, the technique amounts to constructing random variables $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ in Euclidean space which, on the one hand, satisfy the linear dependence $\sum 2^i \mathbf{x}_i \equiv \mathbf{z}$ for a suitably fixed vector $\mathbf{z}$ and, on the other hand, in expectation look independent to any low-degree polynomial $p \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. We pass, then, from p to a univariate polynomial by observing that $\mathbf{E}[p(\mathbf{x}_1, \ldots, \mathbf{x}_n)] = q(\mathbf{z})$ for some univariate polynomial q of degree no greater than the degree of p.

Second, we are able to prove that the rational approximation of the sign function has a self-reducibility property on the discrete domain. More specifically, we are able to give an explicit solution to the *dual* of the rational approximation problem by distributing the nodes as in known positive results. What makes this program possible in the first place is our ability to zero out the dual object on the complementary domain, which is where the above map $M\colon \mathbb{R}[x_1, x_2, \ldots, x_n] \to \mathbb{R}[x]$ plays a crucial role. This dual approach contrasts with previous analyses. In particular, recall that Newman's analysis is specialized to the continuous domain and does not extend to the setting of Theorem 1.7, let alone Theorem 1.6.

Recent progress. A recent follow-up paper [40] proves that the intersection of two halfspaces on $\{0,1\}^n$ has threshold degree $\Theta(n)$, improving on the lower bound of $\Omega(\sqrt{n})$ in this work. We have also learned that the inequality $\deg_\epsilon(F(f, \ldots, f)) \geqslant \deg_\epsilon(F) \deg_\pm(f)$ was derived independently by Lee [30] in a recent work on read-once Boolean formulas.

Organization. Relevant technical preliminaries are provided in Section 2. The main development starts in Section 3, with direct sum theorems for representations of Boolean functions by real polynomials. Sections 4 and 5 study the rational approximation of the canonical halfspace and the majority function, respectively. Our main results on the threshold degree are proved in Section 6, which brings together the results of the preceding sections. Readers who are mainly interested in intersections of halfspaces can safely skip Sections 3.1, 3.4, and 3.5.

2. Preliminaries. Throughout this work, the symbol t refers to a real variable, whereas u, v, w, x, y, z refer to vectors in $\mathbb{R}^n$ and in particular in $\{-1,+1\}^n$. We adopt the following standard definition of the sign function:

$$\operatorname{sgn} t = \begin{cases} -1, & t < 0, \\ 0, & t = 0, \\ 1, & t > 0. \end{cases}$$

We will also have occasion to use the following modified sign function:

$$\widetilde{\operatorname{sgn}}\, t = \begin{cases} -1, & t \leqslant 0, \\ 1, & t > 0. \end{cases}$$

Equations and inequalities involving vectors in $\mathbb{R}^n$, such as $x < y$ or $x \geqslant 0$, are to be interpreted component-wise, as usual.

Throughout this manuscript, we view Boolean functions as mappings $f\colon X \to \{-1,+1\}$ for some finite set X, where -1 and $+1$ correspond to "true" and "false," respectively. In this notation, the conjunction operator becomes $-1 \wedge -1 = -1$, $-1 \wedge 1 = 1 \wedge -1 = 1 \wedge 1 = 1$, and analogously for disjunction. If $\mu_1, \ldots, \mu_k$ are probability distributions on finite sets $X_1, \ldots, X_k$, respectively, then $\mu_1 \times \cdots \times \mu_k$ stands for the probability distribution on $X_1 \times \cdots \times X_k$ given by

$$(\mu_1 \times \cdots \times \mu_k)(x_1, \ldots, x_k) = \prod_{i=1}^{k} \mu_i(x_i).$$

The majority function on n bits, $\mathrm{MAJ}_n\colon \{-1,+1\}^n \to \{-1,+1\}$, is given by

$$\mathrm{MAJ}_n(x) = \widetilde{\mathrm{sgn}}(\textstyle\sum x_i).$$

The symbol P_k stands for the family of all univariate real polynomials of degree up to k. The following combinatorial identity is well-known.

FACT 2.1. *For every integer $n \geqslant 1$ and every polynomial $p \in P_{n-1}$,*

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i p(i) = 0.$$

This fact can be verified by repeated differentiation of the real function

$$(t-1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} t^i$$

at $t = 1$, as explained in [34].

For a real function f on a finite set X, we write $\|f\|_\infty = \max_{x \in X} |f(x)|$. For a subset $X \subseteq \mathbb{R}^n$, we adopt the notation $-X = \{-x : x \in X\}$. We say that a set $X \subseteq \mathbb{R}^n$ is *closed under negation* if $X = -X$. Given a function $f\colon X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ is closed under negation, we say that f is *odd* (respectively, *even*) if $f(-x) = -f(x)$ for all $x \in X$ (respectively, $f(-x) = f(x)$ for all $x \in X$).

Given functions $f\colon X \to \{-1,+1\}$ and $g\colon Y \to \{-1,+1\}$, recall that the function $f \wedge g\colon X \times Y \to \{-1,+1\}$ is given by $(f \wedge g)(x,y) = f(x) \wedge g(y)$. The function $f \vee g$ is defined analogously. Observe that in this notation, $f \wedge f$ and f are completely different functions, the former having domain $X \times X$ and the latter X. These conventions extend in the obvious way to any number of functions. For example, $f_1 \wedge f_2 \wedge \cdots \wedge f_k$ is a Boolean function with domain $X_1 \times X_2 \times \cdots \times X_k$, where X_i is the domain of f_i. Generalizing further, we let the symbol $F(f_1, \ldots, f_k)$ denote the Boolean function on $X_1 \times X_2 \times \cdots \times X_k$ obtained by composing a given function $F\colon \{-1,+1\}^k \to \{-1,+1\}$ with the functions $f_1, f_2, \ldots, f_k$. Finally, recall that the negated function $\overline{f}\colon X \to \{-1,+1\}$ is given by $\overline{f}(x) = -f(x)$.

2.1. Sign-representation and approximation by polynomials. By the *degree* of a multivariate polynomial p on $\mathbb{R}^n$, denoted $\deg p$, we shall always mean the total degree of p, i.e., the greatest total degree of any monomial of p. The degree of a rational function $p(x)/q(x)$ is the maximum of $\deg p$ and $\deg q$. Given a function $f\colon X \to \{-1,+1\}$, where $X \subset \mathbb{R}^n$ is a finite set, the *threshold degree* $\deg_\pm(f)$ of f is defined as the least degree of a multivariate polynomial p such that $f(x)p(x) > 0$ for all $x \in X$. In words, the threshold degree of f is the least degree of a polynomial that represents f in sign. Equivalent terms in the literature include "strong degree" [7], "voting polynomial degree" [27], "polynomial threshold function degree" [35], and "sign degree" [13]. Crucial to understanding the threshold degree is the following result, which is a well-known corollary to Gordan's transposition theorem [17].

THEOREM 2.2 (Gordan). *Let $X \subset \mathbb{R}^n$ be a finite set, $f\colon X \to \{-1,+1\}$ a given function. Then $\deg_\pm(f) > d$ if and only if there exists a probability distribution μ on X such that*

$$\sum_{x\in X} \mu(x)f(x)p(x) = 0$$

for every polynomial p of degree up to d. Equivalently, $\deg_\pm(f) > d$ if and only if there exists a map $\psi\colon X \to \mathbb{R}$, $\psi \not\equiv 0$, such that $f(x)\psi(x) \geqslant 0$ on X and

$$\sum_{x\in X} \psi(x)p(x) = 0$$

for every polynomial p of degree up to d.

Theorem 2.2 has a short proof using linear programming duality, cf. [42, Sec. 2.2].

The threshold degree is closely related to another analytic notion. Let $f\colon X \to \{-1,+1\}$ be given, for a finite subset $X \subset \mathbb{R}^n$. The *ϵ-approximate degree* of f, denoted $\deg_\epsilon(f)$, is the least degree of a polynomial p such that $|f(x)-p(x)| \leqslant \epsilon$ for all $x \in X$. The relationship between the threshold degree and approximate degree is an obvious one:

$$\deg_\pm(f) = \lim_{\epsilon\nearrow 1} \deg_\epsilon(f). \tag{2.1}$$

We will need the following dual characterization of the approximate degree.

THEOREM 2.3. *Fix $\epsilon \geqslant 0$. Let $f\colon X \to \{-1,+1\}$ be given, $X \subset \mathbb{R}^n$ a finite set. Then $\deg_\epsilon(f) > d$ if and only if there exists a function $\psi\colon X \to \mathbb{R}$ such that*

$$\sum_{x\in X} |\psi(x)| = 1,$$
$$\sum_{x\in X} \psi(x)f(x) > \epsilon,$$

and, for every polynomial p of degree up to d,

$$\sum_{x\in X} \psi(x)p(x) = 0.$$

Theorem 2.3 follows readily from linear programming duality, cf. [41, Sec. 3]. Theorem 2.2 can be derived from Theorem 2.3 in view of (2.1).

2.2. Approximation by rational functions. Consider a function $f\colon X \to \{-1,+1\}$, where $X \subseteq \mathbb{R}^n$ is an arbitrary set. For $d \geqslant 0$, we define

$$R(f,d) \;=\; \inf_{p,q} \sup_{x\in X} \left| f(x) - \frac{p(x)}{q(x)} \right|,$$

where the infimum is over multivariate polynomials p and q of degree up to d such that q does not vanish on X. In words, $R(f,d)$ is the least error in an approximation of f by a multivariate rational function of degree up to d. We will also take an interest in the related quantity

$$R^+(f,d) \;=\; \inf_{p,q}\, \sup_{x\in X} \left| f(x) - \frac{p(x)}{q(x)} \right|,$$

where the infimum is over multivariate polynomials p and q of degree up to d such that q is positive on X. These two quantities are related in a straightforward way:

$$R^+(f,2d) \leqslant R(f,d) \leqslant R^+(f,d). \tag{2.2}$$

The second inequality here is trivial. The first follows from the fact that every rational approximant $p(x)/q(x)$ of degree d gives rise to a degree-$2d$ rational approximant with the same error and a positive denominator, namely, $\{p(x)q(x)\}/q(x)^2$. The infimum in the definitions of $R(f,d)$ and $R^+(f,d)$ cannot in general be replaced by a minimum [38], even when X is a finite subset of $\mathbb{R}$. This is in contrast to the more familiar setting of a finite-dimensional normed linear space, where least-error approximants are guaranteed to exist.

We now recall Newman's classical construction [32] of a rational approximant to the sign function.

THEOREM 2.4 (Newman). *Fix $N > 1$. Then for every integer $k \geqslant 1$, there is a rational function $S(t)$ of degree k such that*

$$\max_{1\leqslant|t|\leqslant N} |\operatorname{sgn} t - S(t)| \leqslant 1 - N^{-1/k} \tag{2.3}$$

and the denominator of S is positive on $[-N,-1]\cup[1,N]$.

Proof. (Adapted from Newman [32].) Consider the univariate polynomial

$$p(t) = \prod_{i=1}^{k} \left(t + N^{(2i-1)/(2k)}\right).$$

By examining every interval $[N^{i/(2k)}, N^{(i+1)/(2k)}]$, where $i = 0,1,\dots,2k-1$, one sees that

$$p(t) \geqslant \frac{N^{1/(2k)}+1}{N^{1/(2k)}-1}\,|p(-t)|, \qquad 1 \leqslant t \leqslant N. \tag{2.4}$$

Let

$$S(t) = N^{-1/(2k)} \cdot \frac{p(t)-p(-t)}{p(t)+p(-t)}.$$

Since the functions sgn and S are both odd, it suffices to prove (2.3) for $1 \leqslant t \leqslant N$. Here, we have:

$$\begin{aligned} |1-S(t)| &= \left| 1 - N^{-1/(2k)} \cdot \frac{1-\{p(-t)/p(t)\}}{1+\{p(-t)/p(t)\}} \right| \\ &\leqslant \max\left\{ \left| 1 - N^{-1/(2k)} \cdot \frac{1-\xi}{1+\xi} \right| : |\xi| \leqslant \frac{N^{1/(2k)}-1}{N^{1/(2k)}+1} \right\} \\ &= 1 - N^{-1/k}, \end{aligned}$$

where the second step follows by (2.4). Finally, the positivity of the denominator of S on $[-N,-1]\cup[1,N]$ is immediate from (2.4). ☐

A useful consequence of Newman's theorem is the following general statement on decreasing the error in rational approximation.

THEOREM 2.5. *Let $f\colon X\to\{-1,+1\}$ be given, where $X\subseteq\mathbb{R}^n$. Let d be a given integer, $\epsilon=R(f,d)$. Then for $k=1,2,3,\dots,$*

$$R(f,kd)\leqslant 1-\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/k}.$$

Proof. We may assume that $\epsilon<1$, the theorem being trivial otherwise. Let S be the degree-k rational approximant to the sign function for $N=(1+\epsilon)/(1-\epsilon)$, as constructed in Theorem 2.4. Let $A_1,A_2,\dots,A_m,\dots$ be a sequence of rational functions on X of degree at most d such that $\sup_X|f-A_m|\to\epsilon$ as $m\to\infty$. The theorem follows by considering the sequence of approximants $S(A_m(x)/\{1-\epsilon\})$ as $m\to\infty$. ☐

2.3. Symmetrization. Let S_n denote the symmetric group on n elements. For $\sigma\in S_n$ and $x\in\mathbb{R}^n$, we denote $\sigma x=(x_{\sigma(1)},\dots,x_{\sigma(n)})\in\mathbb{R}^n$. The following is a generalized form of Minsky and Papert's *symmetrization argument* [31], as formulated in [37].

PROPOSITION 2.6 (cf. Minsky and Papert). *Let $n_1,\dots,n_k$ be positive integers. Let $\phi\colon\{0,1\}^{n_1}\times\cdots\times\{0,1\}^{n_k}\to\mathbb{R}$ be a polynomial of degree d. Then there is a polynomial p on $\mathbb{R}^k$ of degree at most d such that for all x in the domain of ϕ,*

$$\mathop{\mathbf{E}}_{\sigma_1\in S_{n_1},\dots,\sigma_k\in S_{n_k}}\big[\phi\big(\sigma_1x_1,\dots,\sigma_kx_k\big)\big]=p\big(\dots,x_{i,1}+\cdots+x_{i,n_i},\dots\big).$$

We now obtain a form of the symmetrization argument for rational approximation.

PROPOSITION 2.7. *Let $n_1,\dots,n_k$ be positive integers, and α,β distinct reals. Let $G\colon\{\alpha,\beta\}^{n_1}\times\cdots\times\{\alpha,\beta\}^{n_k}\to\{-1,+1\}$ be a function such that $G(x_1,\dots,x_k)\equiv G(\sigma_1x_1,\dots,\sigma_kx_k)$ for all $\sigma_1\in S_{n_1},\dots,\sigma_k\in S_{n_k}$. Let d be a given integer. Then for each $\epsilon>R^+(G,d)$, there exists a rational function p/q on $\mathbb{R}^k$ of degree at most d such that for all x in the domain of G, one has*

$$\left|G(x)-\frac{p(\dots,x_{i,1}+\cdots+x_{i,n_i},\dots)}{q(\dots,x_{i,1}+\cdots+x_{i,n_i},\dots)}\right|<\epsilon$$

and $q(\dots,x_{i,1}+\cdots+x_{i,n_i},\dots)>0$.

Proof. Clearly, we may assume that $\epsilon<1$. Using the linear bijection $(\alpha,\beta)\leftrightarrow(0,1)$ if necessary, we may further assume that $\alpha=0$ and $\beta=1$. Since $\epsilon>R^+(G,d)$, there are polynomials P,Q of degree up to d such that for all x in the domain of G, one has $Q(x)>0$ and

$$(1-\epsilon)Q(x)<G(x)P(x)<(1+\epsilon)Q(x).$$

By Proposition 2.6, there exist polynomials p,q on $\mathbb{R}^k$ of degree at most d such that

$$\mathop{\mathbf{E}}_{\sigma_1\in S_{n_1},\dots,\sigma_k\in S_{n_k}}\big[P\big(\sigma_1x_1,\dots,\sigma_kx_k\big)\big]=p\big(\dots,x_{i,1}+\cdots+x_{i,n_i},\dots\big)$$

and

$$\mathop{\mathbf{E}}_{\sigma_1 \in S_{n_1}, \dots, \sigma_k \in S_{n_k}} \big[Q\big(\sigma_1 x_1, \dots, \sigma_k x_k\big)\big] = q\big(\,\dots, x_{i,1} + \dots + x_{i,n_i}, \dots\big)$$

for all x in the domain of G. Then the required properties of p and q follow immediately from the corresponding properties of P and Q. ∎

3. Direct sum theorems. In the several subsections that follow, we prove our direct sum theorems for polynomial representations of composed Boolean functions. General compositions are treated in Section 3.1, followed by conjunctions and other specific compositions in Sections 3.2–3.5.

3.1. General compositions. We begin with general compositions of the form $F(f_1, \dots, f_k)$. This section focuses on results that depend only on the threshold or approximate degrees of $F, f_1, \dots, f_k$. In later sections, we will exploit additional structure of the functions involved. The following result settles Theorems 1.1 and 1.2 from the Introduction.

THEOREM 3.1. *Let $f\colon X \to \{-1,+1\}$ and $F\colon \{-1,+1\}^k \to \{-1,+1\}$ be given functions, where $X \subset \mathbb{R}^n$ is a finite set. Then for $0 < \epsilon < 1$,*

$$\deg_\epsilon(F(f, \dots, f)) \geqslant \deg_\epsilon(F)\deg_\pm(f). \tag{3.1}$$

In particular,

$$\deg_\pm(F(f, \dots, f)) \geqslant \deg_\pm(F)\deg_\pm(f). \tag{3.2}$$

Our proof is inspired by an earlier work of O'Donnell and Servedio [34], who obtained (3.2) for the special case $F = \mathrm{PARITY}$. To prove the general result above, we use dual objects different from those in [34].

Proof of Theorem 3.1. Recall that the threshold degree is a limiting case of the approximate degree, as given by (2.1). Hence, one obtains (3.2) by letting $\epsilon \nearrow 1$ in (3.1). In the remainder of the proof, we focus on (3.1) alone.

Put $D = \deg_\epsilon(F)$ and $d = \deg_\pm(f)$. By Theorem 2.3, a map $\Psi\colon \{-1,+1\}^k \to \mathbb{R}$ exists such that

$$\sum_{z \in \{-1,+1\}^k} |\Psi(z)| = 1, \tag{3.3}$$

$$\sum_{z \in \{-1,+1\}^k} \Psi(z) F(z) > \epsilon, \tag{3.4}$$

and $\sum \Psi(z) p(z) = 0$ for every polynomial p of degree less than D. By Theorem 2.2, there exists a distribution μ on X such that $\sum_X f(x)\mu(x)p(x) = 0$ for every polynomial p of degree less than d.

Now, define $\zeta\colon X^k \to \mathbb{R}$ by

$$\zeta(\dots, x_i, \dots) = 2^k \Psi(\dots, f(x_i), \dots) \prod_{i=1}^k \mu(x_i).$$

We claim that

$$\sum_{X^k} \zeta(\dots, x_i, \dots) p(\dots, x_i, \dots) = 0 \tag{3.5}$$

for every polynomial p of degree less than Dd. By linearity, it suffices to consider a polynomial p of the form $p(\dots,x_i,\dots)=\prod p_i(x_i)$, where $\sum \deg p_i < Dd$. Since Ψ is orthogonal on $\{-1,+1\}^k$ to all polynomials of degree less than D, we have the representation

$$\Psi(z)=\sum_{\substack{S\subseteq\{1,\dots,k\},\\ |S|\geqslant D}} \hat\Psi(S)\prod_{i\in S} z_i$$

for some reals $\hat\Psi(S)$, whence

$$\zeta(\dots,x_i,\dots)=2^k\sum_{\substack{S\subseteq\{1,\dots,k\},\\ |S|\geqslant D}} \hat\Psi(S)\prod_{i\in S} f(x_i)\mu(x_i)\cdot\prod_{i\notin S}\mu(x_i).$$

As a result,

$$\begin{aligned}&\sum_{X^k}\zeta(\dots,x_i,\dots)p(\dots,x_i,\dots)\\ &\qquad=2^k\sum_{|S|\geqslant D}\hat\Psi(S)\prod_{i\in S}\underbrace{\left(\sum_{x_i\in X} f(x_i)\mu(x_i)p_i(x_i)\right)}\prod_{i\notin S}\left(\sum_{x_i\in X}\mu(x_i)p_i(x_i)\right).\end{aligned}\tag{3.6}$$

Since $\sum\deg p_i < Dd$, the pigeonhole principle implies that $\deg p_i < d$ for more than $k-D$ indices $i\in\{1,\dots,k\}$. As a result, for each set S in the outer summation of (3.6), at least one of the underbraced factors vanishes (recall that f is orthogonal on X with respect to μ to all polynomials of degree less than d). This gives (3.5).

We may assume that f is not a constant function, the theorem being trivial otherwise. It follows that $\deg_\pm(f)\geqslant 1$ and $\sum_X f(x)\mu(x)=0$. Now, define a product distribution λ on X^k by $\lambda(\dots,x_i,\dots)=\prod\mu(x_i)$. Since $\sum_X f(x)\mu(x)=0$, it follows that the string $(\dots,f(x_i),\dots)$ is distributed uniformly on $\{-1,+1\}^k$ when $(\dots,x_i,\dots)\sim\lambda$. As a result,

$$\begin{aligned}\sum_{X^k}|\zeta(\dots,x_i,\dots)| &= 2^k\sum_{X^k}|\Psi(\dots,f(x_i),\dots)|\prod_{i=1}^k\mu(x_i)\\ &=2^k\mathop{\mathbf{E}}_{x\sim\lambda}[|\Psi(\dots,f(x_i),\dots)|]\\ &=2^k\mathop{\mathbf{E}}_{z\in\{-1,+1\}^k}[|\Psi(\dots,z_i,\dots)|]\\ &=1,\end{aligned}\tag{3.7}$$

where the last equality holds by (3.3). Similarly,

$$\begin{aligned}\sum_{X^k}\zeta(\dots,x_i,\dots)F(\dots,f(x_i),\dots) &= 2^k\mathop{\mathbf{E}}_{x\sim\lambda}[\Psi(\dots,f(x_i),\dots)F(\dots,f(x_i),\dots)]\\ &=2^k\mathop{\mathbf{E}}_{z\in\{-1,+1\}^k}[\Psi(\dots,z_i,\dots)F(\dots,z_i,\dots)]\\ &>\epsilon,\end{aligned}\tag{3.8}$$

where the inequality holds by (3.4). Now (3.1) follows from (3.5), (3.7), (3.8), and Theorem 2.3. $\square$

REMARK. In Theorem 3.1 and elsewhere in this paper, we consider Boolean functions on finite subsets of $\mathbb{R}^n$, which is the setting of primary interest in computational complexity. Approximation and sign-representation problems on compact infinite sets and other well-behaved sets reduce easily to the finite case [15, Chap. 1, Sec. 5].

We now consider the so-called AND-OR tree, given by (1.1). We improve the standing lower bound on its approximate degree from $\Omega(n^{0.66\cdots})$ to $\Omega(n^{0.75})$, the best upper bound being $O(n)$.

THEOREM 1.3 (restated). *Let* $f\colon \{-1,+1\}^{n^2} \to \{-1,+1\}$ *be given by* $f(x) = \bigvee_{i=1}^n \bigwedge_{j=1}^n x_{ij}$. *Then*

$$\deg_{1/3}(f) = \Omega(n^{0.75}).$$

Proof. Without loss of generality, assume that $n = 4m^2$ for some integer m. Define $g\colon \{-1,+1\}^{4m^3} \to \{-1,+1\}$ by

$$g(x) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{4m^2} x_{ij}.$$

Let $G\colon \{-1,+1\}^{4m} \to \{-1,+1\}$ be given by $G(x) = x_1 \vee \cdots \vee x_{4m}$. A well-known result of Minsky and Papert [31] states that $\deg_\pm(g) = m$. Also, Nisan and Szegedy [33] proved that $\deg_{1/3}(G) = \Theta(\sqrt{m})$. Since $f = G(g,\ldots,g)$, it follows by Theorem 3.1 that $\deg_{1/3}(f) = \Omega(m\sqrt{m})$, as desired. $\square$

We now develop the ideas of Theorem 3.1 to obtain a more general result. For a subset $S \subseteq \{1,2,\ldots,k\}$, its *characteristic vector* $\mathbf{1}_S \in \{-1,+1\}^k$ is given by

$$(\mathbf{1}_S)_i = \begin{cases} -1, & i \in S, \\ 1, & \text{otherwise.} \end{cases}$$

The *block sensitivity* $\operatorname{bs}(F)$ of a Boolean function $F\colon \{-1,+1\}^k \to \{-1,+1\}$ is the maximum number of pairwise disjoint subsets $S_1, S_2, S_3, \ldots \subseteq \{1,2,\ldots,k\}$ such that $F(x) \neq F(x \oplus \mathbf{1}_{S_1}) = F(x \oplus \mathbf{1}_{S_2}) = F(x \oplus \mathbf{1}_{S_3}) = \cdots$ for some $x \in \{-1,+1\}^k$. In words, starting with the string x and flipping the bits in any one set S_i changes the value of the function. It is clear that $0 \leqslant \operatorname{bs}(F) \leqslant k$. It is also easy to verify that the constant functions -1 and 1 have block sensitivity 0, whereas the parity functions $x_1x_2\cdots x_k$ and $-x_1x_2\cdots x_k$ have block sensitivity k. We have:

PROPOSITION 3.2. *Let* $F\colon \{-1,+1\}^k \to \{-1,+1\}$ *be a given Boolean function. Let* $y \in \{-1,+1\}^k$ *be a random string whose* i*th bit is set to* -1 *with probability at most* $\alpha \in [0,1]$, *independently for each* i. *Then for every* $x \in \{-1,+1\}^k$,

$$\mathop{\mathbf{P}}_{y}[F(x_1,\ldots,x_k) \neq F(x_1y_1,\ldots,x_ky_k)] \leqslant \frac{\operatorname{bs}(F)}{\lfloor 1/\alpha \rfloor} \leqslant 2\alpha \operatorname{bs}(F).$$

Proof. Abbreviate $r = \lfloor 1/\alpha \rfloor$. It is clear that

$$\mathop{\mathbf{P}}_{y}[F(x) \neq F(x \oplus y)] \leqslant \mathop{\mathbf{P}}_{y}[F(x) \neq F(x \oplus z) \text{ for some } z \preceq y], \tag{3.9}$$

where $z \preceq y$ is shorthand for $\{i : z_i = -1\} \subseteq \{i : y_i = -1\}$. By monotonicity, it suffices to bound the r.h.s. of (3.9) under the assumption that every bit of y independently takes on -1 with probability $1/r$.

Consider a matrix $Y \in \{-1,+1\}^{r\times n}$ in which each column is chosen independently at random from among $\mathbf{1}_{\{1\}}, \mathbf{1}_{\{2\}}, \dots, \mathbf{1}_{\{r\}}$. Let $y^1, y^2, \dots, y^r$ denote the rows of Y. Although the rows are not independent, each of them *individually* is a random string whose ith bit takes on -1 with probability $1/r$, independently for each i. Hence, the r.h.s. of (3.9) equals

$$\frac{1}{r}\sum_{j=1}^{r} \mathop{\mathbf{P}}_{Y}[F(x) \neq F(x \oplus z) \text{ for some } z \preceq y^j]$$
$$= \frac{1}{r} \mathop{\mathbf{E}}_{Y}\left[\sum_{j=1}^{r} \mathbf{I}[F(x) \neq F(x \oplus z) \text{ for some } z \preceq y^j]\right].$$

The latter summation has at most $\operatorname{bs}(F)$ nonzero terms because $y^1, y^2, \dots, y^r$ are characteristic vectors of pairwise disjoint sets. ∎

We can now state and prove the desired generalization of Theorem 3.1.

THEOREM 3.3. *Let $f\colon X \to \{-1,+1\}$ and $F\colon \{-1,+1\}^k \to \{-1,+1\}$ be given functions, where $X \subset \mathbb{R}^n$ is a finite set. Then for all $\epsilon, \delta > 0$,*

$$\deg_{\epsilon - 4\delta\,\operatorname{bs}(F)}(F(f,\dots,f)) \geqslant \deg_\epsilon(F)\deg_{1-\delta}(f). \tag{3.10}$$

Proof. Let $D = \deg_\epsilon(F)$ and $d = \deg_{1-\delta}(f) > 0$. Theorem 2.3 provides a map $\Psi\colon \{-1,+1\}^k \to \mathbb{R}$ such that

$$\sum_{z\in\{-1,+1\}^k} |\Psi(z)| = 1, \tag{3.11}$$
$$\sum_{z\in\{-1,+1\}^k} \Psi(z)F(z) > \epsilon, \tag{3.12}$$

and $\sum_{z\in\{-1,+1\}^k} \Psi(z)p(z) = 0$ for every polynomial p of degree less than D. Analogously, there exists a map $\psi\colon X \to \mathbb{R}$ such that

$$\sum_{x\in X} |\psi(x)| = 1, \tag{3.13}$$
$$\sum_{x\in X} \psi(x)f(x) > 1-\delta, \tag{3.14}$$

and $\sum_{x\in X} \psi(x)p(x) = 0$ for every polynomial p of degree less than d.

Define $\zeta\colon X^k \to \mathbb{R}$ by

$$\zeta(\dots,x_i,\dots) = 2^k\,\Psi(\dots,\widetilde{\operatorname{sgn}}\,\psi(x_i),\dots)\prod_{i=1}^{k}|\psi(x_i)|.$$

By the same argument as in Theorem 3.1, we have

$$\sum_{X^k} \zeta(\dots,x_i,\dots)p(\dots,x_i,\dots) = 0 \tag{3.15}$$

for every polynomial p of degree less than Dd.

Let λ be the distribution on X^k given by $\lambda(\ldots, x_i, \ldots) = \prod |\psi(x_i)|$. Since ψ is orthogonal to the constant polynomial 1, the string $(\ldots, \widetilde{\operatorname{sgn}}\,\psi(x_i), \ldots)$ is distributed uniformly over $\{-1,+1\}^k$ when one samples $(\ldots, x_i, \ldots)$ according to λ. As a result,

$$\sum_{X^k} |\zeta(\ldots, x_i, \ldots)| = \sum_{z \in \{-1,+1\}^k} |\Psi(z)| = 1, \tag{3.16}$$

where the final equality uses (3.11).

Define

$$\begin{aligned} A_{+1} &= \{x \in X : \psi(x) > 0, f(x) = -1\}, \\ A_{-1} &= \{x \in X : \psi(x) < 0, f(x) = +1\}. \end{aligned}$$

Then

$$\sum_{x \in X} \psi(x) f(x) = \sum_{x \notin A_{-1} \cup A_{+1}} |\psi(x)| - \sum_{x \in A_{-1} \cup A_{+1}} |\psi(x)| = 1 - 2 \sum_{x \in A_{-1} \cup A_{+1}} |\psi(x)|,$$

where the last step uses (3.13). By (3.14), we conclude that $\sum_{x \in A_{+1}} |\psi(x)| < \delta/2$ and $\sum_{x \in A_{-1}} |\psi(x)| < \delta/2$. Furthermore, since ψ is orthogonal to the constant polynomial 1, it follows from (3.13) that

$$\sum_{x : \psi(x) < 0} |\psi(x)| = \sum_{x : \psi(x) > 0} |\psi(x)| = \frac{1}{2}.$$

Now, for any given $z \in \{-1,+1\}^k$, the following two random variables are identically distributed:

(1) the string $(\ldots, f(x_i), \ldots)$ when one chooses $(\ldots, x_i, \ldots) \sim \lambda$ and conditions on the event that $(\ldots, \widetilde{\operatorname{sgn}}\,\psi(x_i), \ldots) = z$;
(2) the string $(\ldots, y_i z_i, \ldots)$, where $y \in \{-1,+1\}^k$ is a random string whose ith bit independently takes on -1 with probability $2 \sum_{x \in A_{z_i}} |\psi(x)| < \delta$.

Proposition 3.2 now implies that for each $z \in \{-1,+1\}^k$,

$$\left| \mathop{\mathbf{E}}_{\lambda} \Big[F(\ldots, f(x_i), \ldots) \mid (\ldots, \widetilde{\operatorname{sgn}}\,\psi(x_i), \ldots) = z \Big] - F(z) \right| \leqslant 4\delta \operatorname{bs}(F). \tag{3.17}$$

Thus,

$$\begin{aligned} &\sum_{X^k} \zeta(\ldots, x_i \ldots) F(\ldots, f(x_i), \ldots) \\ &\qquad = 2^k \mathop{\mathbf{E}}_{\lambda} \Big[\Psi(\ldots, \widetilde{\operatorname{sgn}}\,\psi(x_i), \ldots) F(\ldots, f(x_i), \ldots) \Big] \\ &\qquad \geqslant \sum_{z \in \{-1,+1\}^k} \Psi(z) F(z) - 4\delta \operatorname{bs}(F) \sum_{z \in \{-1,+1\}^k} |\Psi(z)| \\ &\qquad > \epsilon - 4\delta \operatorname{bs}(F), \end{aligned} \tag{3.18}$$

where the last two inequalities use (3.17), (3.11), and (3.12). In view of Theorem 2.3, the exhibited properties (3.15), (3.16), and (3.18) of ζ force (3.10). ∎

Functions $f\colon \{-1,+1\}^n \to \{-1,+1\}$ are known, such as the ODD-MAX-BIT function of Beigel [9], with threshold degree 1 and $(1-\delta)$-approximate degree $n^{\Omega(1)}$

for δ as small as $\delta = \exp\{-n^{\Omega(1)}\}$. Thus, Theorem 3.3 is considerably stronger than Theorem 3.1. Another advantage of Theorem 3.3 is that the $(1-\delta)$-approximate degree is easier to bound from below than the threshold degree [9, 48], even for δ exponentially small. For δ small, the $(1-\delta)$-approximate degree is essentially equivalent to a notion known as *perceptron weight* [31, 9, 48, 28].

COROLLARY 3.4. *Let $f\colon X \to \{-1,+1\}$ and $F\colon \{-1,+1\}^k \to \{-1,+1\}$ be given functions, where $X \subset \mathbb{R}^n$ is a finite set. Then*

$$\deg_{1/3}(F(f,\dots,f)) \geqslant \deg_{1/3}(F)\deg_{1/3}(f)\cdot\Omega\left(\frac{1}{1+\operatorname{bs}(F)}\right).$$

Proof. Letting $\epsilon = 2/3$ and $\delta = 1/(12\operatorname{bs}(F))$ in Theorem 3.3 yields

$$\begin{aligned}\deg_{1/3}(F(f,\dots,f)) &\geqslant \deg_{2/3}(F)\deg_{1-(12\operatorname{bs}(F))^{-1}}(f)\\ &\geqslant \deg_{1/3}(F)\deg_{1/3}(f)\cdot\Omega\left(\frac{1}{1+\operatorname{bs}(F)}\right),\end{aligned}$$

where the second step uses the well-known fact [38] that the sign function can be approximated on $[-1,-\xi]\cup[\xi,1]$ to any constant accuracy by a polynomial of degree $O(1/\xi)$. ∎

We now complement Theorems 3.1 and 3.3 with an *upper* bound for the approximation of composed functions. The following theorem is a refinement of a result due to Buhrman et al. [12], who studied the approximation of Boolean functions with perturbed inputs. The result in this paper assumes low block sensitivity, whereas the development in [12] makes the stronger assumption of low *certificate complexity.*

THEOREM 3.5. *Fix functions $F\colon \{-1,+1\}^k \to \{-1,+1\}$ and $f\colon X \to \{-1,+1\}$, where $X \subset \mathbb{R}^n$ is finite. Then for all $\epsilon, \delta \geqslant 0$,*

$$\deg_{\epsilon+4\delta\operatorname{bs}(F)}(F(f,\dots,f)) \leqslant \deg_\epsilon(F)\deg_\delta(f).$$

Proof. We will follow the approach of Buhrman et al. [12] for most of this proof, with a modest divergence at the end. Fix polynomials P and p on $\{-1,+1\}^k$ and X, respectively. As usual, P may be assumed to be multilinear in view of its domain. Define $\Phi\colon X^k \to \mathbb{R}$ by

$$\Phi(\dots,x_i,\dots) = P\left(\dots,\frac{1}{1+\|f-p\|_\infty}p(x_i),\dots\right).$$

Fix any input $(\dots,x_i,\dots) \in X^k$ and consider a random variable $y \in \{-1,+1\}^k$ whose ith bit takes on -1 with probability

$$\frac{1}{2} - \frac{f(x_i)p(x_i)}{2(1+\|f-p\|_\infty)} \leqslant \|f-p\|_\infty,$$

independently for each i. Then

$$\begin{aligned}
&|\Phi(\dots,x_i,\dots) - F(\dots,f(x_i),\dots)| \\
&\qquad = \left| P\left(\dots, \mathop{\mathbf{E}}_{y_i}[y_i] f(x_i), \dots\right) - F(\dots,f(x_i),\dots)\right| \\
&\qquad = \left| \mathop{\mathbf{E}}_{y}[P(\dots,y_i f(x_i),\dots) - F(\dots,f(x_i),\dots)]\right| \\
&\qquad \leqslant \|P-F\|_\infty + \left| \mathop{\mathbf{E}}_{y}[F(\dots,y_i f(x_i),\dots) - F(\dots,f(x_i),\dots)]\right| \\
&\qquad \leqslant \|P-F\|_\infty + 4\|f-p\|_\infty \operatorname{bs}(F),
\end{aligned}$$

where the second and last steps in the derivation follow by the multilinearity of P and by Proposition 3.2, respectively. ◻

Compositions with k distinct functions. We now consider compositions of the form $F(f_1,\dots,f_k)$, where the functions $f_1,\dots,f_k$ may all be distinct. For a function $F\colon \{-1,+1\}^k \to \mathbb{R}$ and a vector $v=(v_1,\dots,v_k)$ of nonnegative reals, define the (ϵ,v)-*approximate degree* $\deg_{\epsilon,v}(F)$ to be the least D for which there is a polynomial $P(x_1,\dots,x_k)$ with

$$P \in \operatorname{span}\left\{ \prod_{i\in S} x_i : S\subseteq\{1,2,\dots,k\},\ \sum_{i\in S} v_i \leqslant D\right\}$$

and $\|F-P\|_\infty \leqslant \epsilon$. Note that the ϵ-approximate degree of F is the (ϵ,v)-approximate degree of F for $v=(1,1,\dots,1)$. It is straightforward to see that $\deg_{\epsilon,v}(F)\geqslant \deg_{\epsilon,u}(F)$ for $v\geqslant u$. It is also clear that

$$\deg_{\epsilon,v}(F) \geqslant \min_{i_1<i_2<\dots<i_{\deg_\epsilon(F)}} \{v_{i_1}+v_{i_2}+\dots+v_{i_{\deg_\epsilon(F)}}\},$$

since any polynomial that approximates F within ϵ must feature a monomial of degree $\deg_\epsilon(F)$. We will need the following generalized version of Theorem 2.3, due to Ioffe and Tikhomirov [19].

THEOREM 3.6 (Ioffe and Tikhomirov). *Let X be a finite set, $\epsilon>0$. Fix any family Φ of functions $X\to\mathbb{R}$ and an additional function $f\colon X\to\mathbb{R}$. Then*

$$\min_{\phi\in\operatorname{span}(\Phi)} \|f-\phi\|_\infty > \epsilon$$

if and only if there exists a function $\psi\colon X\to\mathbb{R}$ in the orthogonal complement of $\operatorname{span}(\Phi)$ *such that*

$$\sum_{x\in X}|\psi(x)| = 1,$$
$$\sum_{x\in X}\psi(x)f(x) > \epsilon.$$

A short proof of Theorem 3.6 can be found, e.g., in [41, Sec. 3]. With this setup in place, we obtain analogues of Theorems 3.3 and 3.5 for compositions of the form $F(f_1,\dots,f_k)$.

THEOREM 3.7. *Fix nonconstant functions $f_i\colon X_i\to\{-1,+1\}$ $(i=1,2,\dots,k)$ and $F\colon\{-1,+1\}^k\to\{-1,+1\}$, where each $X_i\subset\mathbb{R}^n$ is finite. Then for all $\epsilon,\delta>0$,* *one has*

$$\deg_{\epsilon-4\delta\,\mathrm{bs}(F)}(F(f_1,\dots,f_k)) \geqslant \deg_{\epsilon,v}(F),$$

where $v = (\deg_{1-\delta}(f_1),\dots,\deg_{1-\delta}(f_k))$.

Proof. Let $D = \deg_{\epsilon,v}(F)$. Define $\Phi = \{\chi_S : \sum_{i\in S} v_i < D\}$, where each $\chi_S\colon \{-1,+1\}^k \to \mathbb{R}$ is given by $\chi_S(z) = \prod_{i\in S} z_i$. Note that $\operatorname{span}(\Phi)$ has orthogonal complement $\operatorname{span}\{\chi_S : \sum_{i\in S} v_i \geqslant D\}$. By definition,

$$\max_{\phi\in\operatorname{span}(\Phi)} \|F-\phi\|_\infty > \epsilon.$$

Applying Theorem 3.6 with Φ and $f = F$ gives a map $\Psi\colon \{-1,+1\}^k \to \mathbb{R}$ with

$$\sum_{z\in\{-1,+1\}^k} |\Psi(z)| = 1, \tag{3.19}$$

$$\sum_{z\in\{-1,+1\}^k} \Psi(z)F(z) > \epsilon,$$

and

$$\Psi(z) = \sum_{S\in\mathscr{S}} \hat\Psi(S) \prod_{i\in S} z_i$$

for some reals $\hat\Psi(S)$, where $\mathscr{S} = \{S \subseteq \{1,2,\dots,k\} : \sum_{i\in S} v_i \geqslant D\}$. Analogously, there are maps $\psi_i\colon X_i \to \mathbb{R}$, $i = 1,2,\dots,k$, such that

$$\sum_{x_i\in X_i} |\psi_i(x_i)| = 1,$$

$$\sum_{x_i\in X_i} \psi_i(x_i)f_i(x_i) > 1-\delta,$$

and $\sum_{x_i\in X_i} \psi_i(x_i)p(x_i) = 0$ for every polynomial p of degree less than v_i.

Define $\zeta\colon X_1\times\cdots\times X_k \to \mathbb{R}$ by

$$\zeta(\dots,x_i,\dots) = 2^k\,\Psi(\dots,\widetilde{\operatorname{sgn}}\,\psi_i(x_i),\dots)\prod_{i=1}^k |\psi_i(x_i)|.$$

By an argument analogous to that in Theorem 3.1, we have

$$\sum_{X_1\times\cdots\times X_k} \zeta(\dots,x_i,\dots)p(\dots,x_i,\dots) = 0 \tag{3.20}$$

for every polynomial p of degree less than D.

Let λ be the probability distribution on $X_1\times\cdots\times X_k$ given by $\lambda(\dots,x_i,\dots) = \prod |\psi_i(x_i)|$. Since each ψ_i is orthogonal to the constant polynomial 1, the string $(\dots,\widetilde{\operatorname{sgn}}\,\psi_i(x_i),\dots)$ is distributed uniformly over $\{-1,+1\}^k$ when $(\dots,x_i,\dots)\sim\lambda$. As a result,

$$\sum_{X_1\times\cdots\times X_k} |\zeta(\dots,x_i,\dots)| = \sum_{z\in\{-1,+1\}^k} |\Psi(z)| = 1, \tag{3.21}$$

where the final equality uses (3.19). By an argument analogous to that in Theorem 3.3,

$$\sum_{X_1\times\cdots\times X_k} \zeta(\dots,x_i\dots)F(\dots,f_i(x_i),\dots) > \epsilon - 4\delta\,\mathrm{bs}(F). \tag{3.22}$$

By Theorem 2.3, the exhibited properties (3.20)–(3.22) of ζ complete the proof. ∎

COROLLARY 3.8. *Fix nonconstant functions* $f_i\colon X_i \to \{-1,+1\}$ $(i=1,2,\dots,k)$ *and* $F\colon \{-1,+1\}^k \to \{-1,+1\}$, *where each* $X_i \subset \mathbb{R}^n$ *is finite. Then*

$$\deg_{1/3}(F(f_1,\dots,f_k)) \geqslant \Omega\left(\frac{\deg_{2/3,v}(F)}{1+\mathrm{bs}(F)}\right),$$

where $v = (\deg_{1/3}(f_1),\dots,\deg_{1/3}(f_n))$.

Proof. The proof is entirely analogous to Corollary 3.4. Specifically, letting $\epsilon = 2/3$ and $\delta = 1/(12\,\mathrm{bs}(F)$ in Theorem 3.7 yields

$$\deg_{1/3}(F(f_1,\dots,f_k)) \geqslant \deg_{2/3,u}(F),$$

where $u = (\deg_{1-\delta}(f_1),\dots,\deg_{1-\delta}(f_k))$. This completes the proof since $\deg_{1-\delta}(f_i) \geqslant \Omega(\delta\deg_{1/3}(f_i))$ for each i. ∎

We conclude with a generalization of Theorem 3.5 to the case of distinct functions.

THEOREM 3.9. *Fix Boolean functions* $F\colon \{-1,+1\}^k \to \{-1,+1\}$ *and* $f_i\colon X_i \to \{-1,+1\}$, $i=1,2,\dots,k$, *where each* $X_i \subset \mathbb{R}^n$ *is finite. Then for all* $\epsilon,\delta \geqslant 0$,

$$\deg_{\epsilon+4\delta\,\mathrm{bs}(F)}(F(f_1,\dots,f_k)) \leqslant \deg_{\epsilon,v}(F),$$

where $v = (\deg_\delta(f_1),\dots,\deg_\delta(f_k))$.

Proof. Fix a real polynomial P on $\{-1,+1\}^k$ and polynomials p_i on X_i, respectively. As usual, P may be assumed to be multilinear in view of its domain. Define $\Phi\colon X_1\times\cdots\times X_k \to \mathbb{R}$ by

$$\Phi(\dots,x_i,\dots) = P\left(\dots,\frac{1}{1+\|f_i-p_i\|_\infty}p_i(x_i),\dots\right).$$

The remainder of the proof is analogous to that of Theorem 3.5. ∎

3.2. Auxiliary results on rational approximation. In this section, we prove a number of auxiliary facts about uniform approximation and sign-representation. This preparatory work will set the stage for the analysis of conjunctions of functions. We start by spelling out the exact relationship between the rational approximation and sign-representation of a Boolean function.

THEOREM 3.10. *Let* $f\colon X \to \{-1,+1\}$ *be a given function, where* $X \subset \mathbb{R}^n$ *is finite. Then for every integer* d,

$$\deg_\pm(f) \leqslant d \quad\Leftrightarrow\quad R^+(f,d) < 1.$$

Proof. For the forward implication, let p be a polynomial of degree at most d such that $f(x)p(x) > 0$ for every $x \in X$. Letting $M = \max_{x\in X}|p(x)|$ and $m = \min_{x\in X}|p(x)|$, we have

$$R^+(f,d) \leqslant \max_{x\in X}\left|f(x) - \frac{p(x)}{M}\right| \leqslant 1 - \frac{m}{M} < 1.$$

For the converse, fix a degree-d rational function $p(x)/q(x)$ with $q(x) > 0$ on X and $\max_X |f(x) - \{p(x)/q(x)\}| < 1$. Then clearly $f(x)p(x) > 0$ on X. ∎

Our next observation amounts to reformulating the rational approximation of Boolean functions in a way that is more analytically pleasing.

Theorem 3.11. *Let $f\colon X \to \{-1,+1\}$ be a given function, where $X \subset \mathbb{R}^n$ is finite. Then for every integer $d \geqslant \deg_\pm(f)$, one has*

$$R^+(f,d) = \inf_{c\geqslant 1} \frac{c^2-1}{c^2+1},$$

where the infimum is over all $c \geqslant 1$ for which there exist polynomials p, q of degree up to d such that $0 < \frac{1}{c}q(x) \leqslant f(x)p(x) \leqslant cq(x)$ on X.

Proof. In view of Theorem 3.10, the quantity $R^+(f,d)$ is the infimum over all $\epsilon < 1$ for which there exist polynomials p and q of degree no greater than d such that $0 < (1-\epsilon)q(x) \leqslant f(x)p(x) \leqslant (1+\epsilon)q(x)$ on X. Equivalently, one may require that

$$0 < \frac{1-\epsilon}{\sqrt{1-\epsilon^2}}\, q(x) \leqslant f(x)p(x) \leqslant \frac{1+\epsilon}{\sqrt{1-\epsilon^2}}\, q(x).$$

Letting $c = c(\epsilon) = \sqrt{(1+\epsilon)/(1-\epsilon)}$, the theorem follows. ∎

We will now show that if a degree-d rational approximant achieves error ϵ in approximating a given Boolean function, then a degree-$2d$ approximant can achieve error as small as ϵ^2. Note that this result is a refinement of Theorem 2.5 for small k.

Theorem 3.12. *Let $f\colon X \to \{-1,+1\}$ be a given function, where $X \subseteq \mathbb{R}^n$. Let d be a given integer. Then*

$$R^+(f,2d) \leqslant \left(\frac{\epsilon}{1+\sqrt{1-\epsilon^2}}\right)^2,$$

where $\epsilon = R(f,d)$.

Proof. The theorem is clearly true for $\epsilon = 1$. For $0 \leqslant \epsilon < 1$, consider the univariate rational function

$$S(t) = \frac{4\sqrt{1-\epsilon^2}}{1+\sqrt{1-\epsilon^2}} \cdot \frac{t}{t^2 + (1-\epsilon^2)}.$$

Calculus shows that

$$\max_{1-\epsilon\leqslant|t|\leqslant 1+\epsilon} |\operatorname{sgn} t - S(t)| = \left(\frac{\epsilon}{1+\sqrt{1-\epsilon^2}}\right)^2.$$

Fix a sequence $A_1, A_2, \dots$ of rational functions of degree no greater than d such that $\sup_{x\in X} |f(x) - A_m(x)| \to \epsilon$ as $m \to \infty$. Then $S(A_1(x)), S(A_2(x)), \dots$ is the sought sequence of approximants to f, each a rational function of degree at most $2d$ with a positive denominator. ∎

Corollary 3.13. *Let $f\colon X \to \{-1,+1\}$ be a given function, where $X \subseteq \mathbb{R}^n$. Then for all integers $d \geqslant 1$ and reals $t \geqslant 2$,*

$$R^+(f,td) \leqslant R(f,d)^{t/2}.$$

Proof. If $t = 2^k$ for some integer $k \geqslant 1$, then repeated applications of Theorem 3.12 yield $R^+(f,2^k d) \leqslant R(f,2^{k-1}d)^2 \leqslant \cdots \leqslant R(f,d)^{2^k}$. The general case follows because $2^{\lfloor \log t\rfloor} \geqslant t/2$. ∎

3.3. Conjunctions of functions. In this section, we prove direct sum theorems for conjunctions of Boolean functions. Recall that a key challenge will be, given a sign-representation $\phi(x,y)$ of a composite function $f(x) \wedge g(y)$, to suitably break down ϕ and recover individual rational approximants of f and g. We now present an ingredient of the solution, namely, a certain fact about pairs of matrices based on Farkas' Lemma. For the time being, we will formulate it in a clean and abstract way.

THEOREM 3.14. *Fix matrices $A, B \in \mathbb{R}^{m\times n}$ and a real $c \geqslant 1$. Consider the following system of linear inequalities in $u, v \in \mathbb{R}^n$:*

$$\left.\begin{aligned} \frac{1}{c}\,Au \leqslant\, &Bv \leqslant cAu,\\ &u \geqslant 0,\\ &v \geqslant 0. \end{aligned}\right\} \tag{3.23}$$

If $u = v = 0$ is the only solution to (3.23)*, then there exist vectors $w \geqslant 0$ and $z \geqslant 0$ such that*

$$w^{\mathsf{T}}A + z^{\mathsf{T}}B > c(z^{\mathsf{T}}A + w^{\mathsf{T}}B).$$

Proof. If $u = v = 0$ is the only solution to (3.23), then linear programming duality implies the existence of vectors $w \geqslant 0$ and $z \geqslant 0$ such that $w^{\mathsf{T}}A > cz^{\mathsf{T}}A$ and $z^{\mathsf{T}}B > cw^{\mathsf{T}}B$. Adding the last two inequalities completes the proof. ∎

For clarity of exposition, we first prove the main result of this section for the case of *two* Boolean functions at least one of which is *odd*. While this case seems restricted, we will see that it captures the full complexity of the problem.

THEOREM 3.15. *Let $f\colon X \to \{-1,+1\}$ and $g\colon Y \to \{-1,+1\}$ be given functions, where $X, Y \subset \mathbb{R}^n$ are arbitrary finite sets. Assume that $f \not\equiv 1$ and $g \not\equiv 1$. Let $d = \deg_\pm(f \wedge g)$. If f is odd, then*

$$R^+(f, 2d) + R^+(g, d) < 1.$$

Proof. We first collect some basic observations. Since $f \not\equiv 1$ and $g \not\equiv 1$, we have $\deg_\pm(f) \leqslant d$ and $\deg_\pm(g) \leqslant d$. Therefore, Theorem 3.10 implies that

$$R^+(f, d) < 1, \qquad R^+(g, d) < 1.$$

In particular, the theorem holds if $R^+(g, d) = 0$. In the remainder of the proof, we assume that $R^+(g, d) = \epsilon$, where $0 < \epsilon < 1$.

By hypothesis, there exists a degree-d polynomial ϕ such that $f(x) \wedge g(y) = \operatorname{sgn} \phi(x,y)$ for all $x \in X$, $y \in Y$. Define

$$X^- = \{x \in X : f(x) = -1\}.$$

Since X is closed under negation and f is odd, we have $f(x) = 1 \Leftrightarrow -x \in X^-$. We will make several uses of this fact in what follows, without further mention.

Put

$$c = \sqrt{\frac{1 + (1-\delta)\epsilon}{1 - (1-\delta)\epsilon}},$$

where $\delta = \delta(\epsilon) \in (0,1)$ is sufficiently small. Since $R^+(g,d) > (c^2-1)/(c^2+1)$, we know by Theorem 3.11 that there cannot exist polynomials p,q of degree up to d such that

$$0 < \frac{1}{c}q(y) \leqslant g(y)p(y) \leqslant cq(y), \qquad y \in Y. \tag{3.24}$$

We claim, then, that there cannot exist reals $a_x \geqslant 0$, $x \in X$, not all zero, such that

$$\frac{1}{c}\sum_{x\in X^-} a_{-x}\phi(-x,y) \leqslant g(y)\sum_{x\in X^-} a_x\phi(x,y) \leqslant c\sum_{x\in X^-} a_{-x}\phi(-x,y), \quad y \in Y.$$

Indeed, if such reals a_x were to exist, then (3.24) would hold for the polynomials $p(y) = \sum_{x\in X^-} a_x\phi(x,y)$ and $q(y) = \sum_{x\in X^-} a_{-x}\phi(-x,y)$. In view of the nonexistence of the a_x, Theorem 3.14 applies to the matrices

$$\Big[\phi(-x,y)\Big]_{y\in Y,\, x\in X^-}, \qquad \Big[g(y)\phi(x,y)\Big]_{y\in Y,\, x\in X^-}$$

and guarantees the existence of nonnegative reals λ_y, μ_y for $y \in Y$ such that

$$\begin{aligned}\sum_{y\in Y}\lambda_y\phi(-x,y) &+ \sum_{y\in Y}\mu_y g(y)\phi(x,y)\\ &> c\left(\sum_{y\in Y}\mu_y\phi(-x,y) + \sum_{y\in Y}\lambda_y g(y)\phi(x,y)\right), \qquad x \in X^-.\end{aligned} \tag{3.25}$$

Define polynomials α, β on X by

$$\begin{aligned}\alpha(x) &= \sum_{y\in g^{-1}(-1)} \{\lambda_y\phi(-x,y) - \mu_y\phi(x,y)\},\\ \beta(x) &= \sum_{y\in g^{-1}(1)} \{\lambda_y\phi(-x,y) + \mu_y\phi(x,y)\}.\end{aligned}$$

Then (3.25) can be restated as

$$\alpha(x) + \beta(x) > c\{-\alpha(-x) + \beta(-x)\}, \qquad x \in X^-.$$

Both members of this inequality are nonnegative, which forces $\{\alpha(x)+\beta(x)\}^2 > c^2\{-\alpha(-x)+\beta(-x)\}^2$ for $x \in X^-$. Since in addition $\alpha(-x) \leqslant 0$ and $\beta(-x) \geqslant 0$ for $x \in X^-$, we have

$$\{\alpha(x)+\beta(x)\}^2 > c^2\{\alpha(-x)+\beta(-x)\}^2, \qquad x \in X^-.$$

Letting $\gamma(x) = \{\alpha(x)+\beta(x)\}^2$, we see that

$$R^+(f,2d) \leqslant \max_{x\in X}\left| f(x) - \frac{c^2+1}{c^2}\cdot\frac{\gamma(-x)-\gamma(x)}{\gamma(-x)+\gamma(x)}\right| \leqslant \frac{1}{c^2} < 1-\epsilon,$$

where the final inequality holds for all $\delta = \delta(\epsilon) \in (0,1)$ small enough. $\square$

REMARK. In Theorem 3.15 and elsewhere in this paper, the degree of a multivariate polynomial $p(x_1, x_2, \ldots, x_n)$ is defined as the greatest total degree of any monomial of p. A related notion is the *partial degree* of p, which is the maximum degree of p in any one of the variables $x_1, x_2, \ldots, x_n$. The proof of Theorem 3.15 applies unchanged to this alternate notion. Specifically, if the conjunction $f(x) \wedge g(y)$ can be sign-represented by a polynomial of partial degree d, then there exist rational functions $F(x)$ and $G(y)$ of partial degree $2d$ such that $\|f - F\|_\infty + \|g - G\|_\infty < 1$. In the same way, the program of Section 3.4 carries over, with cosmetic changes, to the notion of partial degree. Other, more abstract notions of degree can be handled.

As promised, we will now remove the assumption, made in Theorem 3.15, about one of the functions being odd. The result that we are about to prove settles Theorem 1.4 from the Introduction.

THEOREM 3.16. *Let $f\colon X \to \{-1, +1\}$ and $g\colon Y \to \{-1, +1\}$ be given functions, where $X, Y \subset \mathbb{R}^n$ are arbitrary finite sets. Assume that $f \not\equiv 1$ and $g \not\equiv 1$. Let $d = \deg_\pm(f \wedge g)$. Then*

$$R^+(f, 4d) + R^+(g, 2d) < 1 \tag{3.26}$$

and, by symmetry,

$$R^+(f, 2d) + R^+(g, 4d) < 1.$$

Proof. It suffices to prove (3.26). Define $X' \subset \mathbb{R}^{n+1}$ by $X' = \{(x, 1), (-x, -1) : x \in X\}$. It is clear that X' is closed under negation. Let $f'\colon X' \to \{-1, +1\}$ be the odd Boolean function given by

$$f'(x, b) = \begin{cases} f(x), & b = 1, \\ -f(-x), & b = -1. \end{cases}$$

Let ϕ be a polynomial of degree no greater than d such that $f(x) \wedge g(y) \equiv \operatorname{sgn} \phi(x, y)$. Fix an input $\tilde{x} \in X$ such that $f(\tilde{x}) = -1$. Then for large enough $K \gg 1$ one has $f'(x, b) \wedge g(y) \equiv \operatorname{sgn}\{K(1 + b)\phi(x, y) + \phi(-x, y)\phi(\tilde{x}, y)\}$, whence

$$\deg_\pm(f' \wedge g) \leqslant 2d.$$

Theorem 3.15 now yields $R^+(f', 4d) + R^+(g, 2d) < 1$. Since $R^+(f, 4d) \leqslant R^+(f', 4d)$ by definition, the proof is complete. ∎

Finally, we obtain an analogue of Theorem 3.16 for a conjunction of three and more functions.

THEOREM 3.17. *Let $f_1, f_2, \ldots, f_k$ be given Boolean functions on finite sets $X_1, X_2, \ldots, X_k \subset \mathbb{R}^n$, respectively. Assume that $f_i \not\equiv 1$ for $i = 1, 2, \ldots, k$. Let $d = \deg_\pm(f_1 \wedge f_2 \wedge \cdots \wedge f_k)$. Then*

$$\sum_{i=1}^{k} R^+(f_i, D) < 1$$

for $D = 8d \log 2k$.

Proof. Since $f_1, f_2, \ldots, f_k \not\equiv 1$, it follows that for each pair of indices $i < j$, the function $f_i \wedge f_j$ is a subfunction of $f_1 \wedge f_2 \wedge \cdots \wedge f_k$. Theorem 3.16 now shows that for each $i < j$,

$$R^+(f_i, 4d) + R^+(f_j, 4d) < 1. \tag{3.27}$$

Without loss of generality, we may assume that $R^+(f_1, 4d) = \max_{i=1,\dots,k}\{R^+(f_i, 4d)\}$. Put $\epsilon = R^+(f_1, 4d)$. By (3.27),

$$R^+(f_i, 4d) < \min\left\{1-\epsilon, \frac{1}{2}\right\}, \qquad i = 2, 3, \dots, k.$$

Now Corollary 3.13 implies that

$$\sum_{i=1}^{k} R^+(f_i, D) \leqslant \epsilon + \sum_{i=2}^{k} R^+(f_i, 4d)^{1+\log k} < 1. \qquad \square$$

3.4. Other combining functions. As we will now see, the development in Section 3.3 applies to many combining functions other than conjunctions. Disjunctions are an obvious starting point. Consider two Boolean functions $f\colon X \to \{-1, +1\}$ and $g\colon Y \to \{-1, +1\}$, where $X, Y \subset \mathbb{R}^n$ are finite sets and $f, g \not\equiv -1$. Let $d = \deg_{\pm}(f \vee g)$. Then, we claim that

$$R^+(f, 4d) + R^+(g, 4d) < 1. \tag{3.28}$$

To see this, note first that the function $f \vee g$ has the same threshold degree as its negation, $\overline{f} \wedge \overline{g}$. Applying Theorem 3.16 to the latter function shows that

$$R^+(\overline{f}, 4d) + R^+(\overline{g}, 4d) < 1.$$

This is equivalent to (3.28) since approximating a function is the same as approximating its negation: $R^+(\overline{f}, 4d) = R^+(f, 4d)$ and $R^+(\overline{g}, 4d) = R^+(g, 4d)$. As in the case of conjunctions, (3.28) can be strengthened to

$$R^+(f, 2d) + R^+(g, 2d) < 1$$

if at least one of f, g is known to be odd. These observations carry over to disjunctions of multiple functions, $f_1 \vee f_2 \vee \cdots \vee f_k$.

We start with some notation and definitions. Let $f, h\colon \{-1, +1\}^k \to \{-1, +1\}$ be given Boolean functions. Recall that f is called a *subfunction* of h if for some fixed strings $y, z \in \{-1, +1\}^k$, one has

$$f(x) = h(\dots, (x_i \wedge y_i) \vee z_i, \dots)$$

for each $x \in \{-1, +1\}^k$. In words, f can be obtained from h by replacing some of the variables $x_1, x_2, \dots, x_k$ with fixed values (-1 or $+1$).

DEFINITION 3.18. A function $F\colon \{-1, +1\}^k \to \{-1, +1\}$ is AND-*reducible* if for each pair of indices i, j, where $1 \leqslant i < j \leqslant k$, at least one of the eight functions

$$\begin{array}{ll} x_i \wedge x_j, & x_i \vee x_j, \\ x_i \wedge \overline{x_j}, & x_i \vee \overline{x_j}, \\ \overline{x_i} \wedge x_j, & \overline{x_i} \vee x_j, \\ \overline{x_i} \wedge \overline{x_j}, & \overline{x_i} \vee \overline{x_j} \end{array}$$

is a subfunction of $F(x)$.

THEOREM 3.19. *Let $f_1, f_2, \dots, f_k$ be nonconstant Boolean functions on finite sets $X_1, X_2, \dots, X_k \subset \mathbb{R}^n$, respectively. Let $F\colon \{-1, +1\}^k \to \{-1, +1\}$ be an* AND-*reducible function. Put $d = \deg_{\pm}(F(f_1, f_2, \dots, f_k))$. Then*

$$\sum_{i=1}^{k} R^+(f_i, D) < 1$$

for $D = 8d\log 2k$.

Proof. Since F is AND-reducible, it follows that for each pair of indices $i < j$, one of the following eight functions is a subfunction of $F(f_1, \dots, f_k)$:

$$\begin{array}{ll} f_i \wedge f_j, & f_i \vee f_j, \\ f_i \wedge \overline{f_j}, & f_i \vee \overline{f_j}, \\ \overline{f_i} \wedge f_j, & \overline{f_i} \vee f_j, \\ \overline{f_i} \wedge \overline{f_j}, & \overline{f_i} \vee \overline{f_j}. \end{array}$$

By Theorem 3.16 (and the opening remarks of this section),

$$R^+(f_i, 4d) + R^+(f_j, 4d) < 1.$$

The remainder of the proof is identical to the proof of Theorem 3.17, starting at equation (3.27). ☐

For a function $F\colon \{-1,+1\}^k \to \{-1,+1\}$ to be AND-reducible, F must clearly depend on all of its inputs. This necessary condition is often sufficient, for example when F is a read-once AND/OR/NOT formula or a halfspace. Hence, Theorem 1.5 from the Introduction is a corollary of Theorem 3.19.

3.5. Additional observations. Analogous to Section 3.1, our results here can be viewed as a technique for proving lower bounds on the threshold degree of composite functions $F(f_1, f_2, \dots, f_k)$. We make this view explicit in the following statement, which is the contrapositive of Theorem 3.19.

THEOREM 3.20. *Let $f_1, f_2, \dots, f_k$ be nonconstant Boolean functions on finite sets $X_1, X_2, \dots, X_k \subset \mathbb{R}^n$, respectively. Let $F\colon \{-1,+1\}^k \to \{-1,+1\}$ be an* AND-*reducible function. Suppose that $\sum R^+(f_i, D) \geqslant 1$ for some integer D. Then*

$$\deg_\pm(F(f_1, f_2, \dots, f_k)) > \frac{D}{8\log 2k}. \tag{3.29}$$

REMARK 3.21 (On the tightness of Theorem 3.20). Theorem 3.20 is close to optimal. For example, when $F = \text{AND}$, the lower bound in (3.29) is tight up to a factor of $\Theta(k\log k)$. This can be seen by the well-known argument [10] described in the Introduction. Specifically, fix an integer D such that $\sum R^+(f_i, D) < 1$. Then there exists a degree-D rational function $p_i(x_i)/q_i(x_i)$ on X_i, for $i = 1, 2, \dots, k$, such that q_i is positive on X_i and

$$\sum_{i=1}^{k} \max_{x_i \in X_i} \left| f_i(x_i) - \frac{p_i(x_i)}{q_i(x_i)} \right| < 1.$$

As a result,

$$\bigwedge_{i=1}^{k} f_i(x_i) \equiv \operatorname{sgn}\left(k - 1 + \sum_{i=1}^{k} f_i(x_i) \right) \equiv \operatorname{sgn}\left(k - 1 + \sum_{i=1}^{k} \frac{p_i(x_i)}{q_i(x_i)} \right).$$

Multiplying by $\prod q_i(x_i)$ yields

$$\bigwedge_{i=1}^{k} f_i(x_i) \equiv \operatorname{sgn}\left((k-1)\prod_{i=1}^{k} q_i(x_i) + \sum_{i=1}^{k} p_i(x_i) \prod_{j \in \{1,\dots,k\}\setminus\{i\}} q_j(x_j) \right),$$

whence $\deg_{\pm}(f_1 \wedge f_2 \wedge \cdots \wedge f_k) \leqslant kD$. This settles the claim regarding $F = \text{AND}$. For arbitrary AND-reducible functions $F\colon \{-1,+1\}^k \to \{-1,+1\}$, a similar argument (cf. Theorem 31 of Klivans et al. [22]) shows that the lower bound in (3.29) is tight up to a polynomial in k.

We close this section with one additional result.

THEOREM 3.22. *Let $f\colon X \to \{-1,+1\}$ be a given function, where $X \subset \mathbb{R}^n$ is finite. Then for every integer $k \geqslant 2$,*

$$\deg_{\pm}(\underbrace{f \wedge f \wedge \cdots \wedge f}_{k}) \leqslant (8k \log k) \cdot \deg_{\pm}(f \wedge f). \tag{3.30}$$

Proof. Put $d = \deg_{\pm}(f \wedge f)$. Theorem 3.16 implies that $R^+(f, 4d) < 1/2$, whence $R^+(f, 8d \log k) < 1/k$ by Corollary 3.13. By the argument in Remark 3.21, this proves the theorem. □

To illustrate, let $\mathscr{C}$ be a given class of functions on $\{-1,+1\}^n$, such as halfspaces. Theorem 3.22 shows that the task of constructing a sign-representation for the intersections of up to k members from $\mathscr{C}$ reduces to the case $k = 2$. In other words, solving the problem for $k = 2$ essentially solves it for all k. The dependence on k in (3.30) is tight up to a factor of $16 \log k$, even in the simple case when f is the OR function [31].

4. Rational approximation of a halfspace. In this section, we determine how well a rational function of any given degree can approximate the canonical halfspace. The lower bounds in Theorem 1.6, the main result to be proved in this section, are considerably more involved than the upper bounds. To help build some intuition in the former case, we first obtain the upper bounds (Section 4.1) and then the lower bounds (Sections 4.2 and 4.3).

4.1. Upper bounds. As shown in the Introduction, the OR function on n bits has $R^+(\text{OR}, 1) = 0$. A similar example is the ODD-MAX-BIT function $f\colon \{0,1\}^n \to \{-1,+1\}$, due to Beigel [9], defined by

$$f(x) = \operatorname{sgn}\left(1 + \sum_{i=1}^{n} (-2)^i x_i\right).$$

Indeed, letting

$$A_M(x) = \frac{1 + \sum_{i=1}^n (-M)^i x_i}{1 + \sum_{i=1}^n M^i x_i},$$

we have $\|f - A_M\|_\infty \to 0$ as $M \to \infty$. Thus, $R^+(f, 1) = 0$. With this construction in mind, we now turn to the canonical halfspace. We start with an auxiliary result that generalizes the argument just given.

LEMMA 4.1. *Let $f\colon \{0, \pm 1, \pm 2\}^n \to \{-1,+1\}$ be the function given by $f(z) = \operatorname{sgn}(1 + \sum_{i=1}^n 2^i z_i)$. Then*

$$R^+(f, 64) = 0.$$

Proof. Consider the deterministic finite automaton in Figure 4.1. The automaton has two terminal states (labeled "+" and "−") and three nonterminal states (the

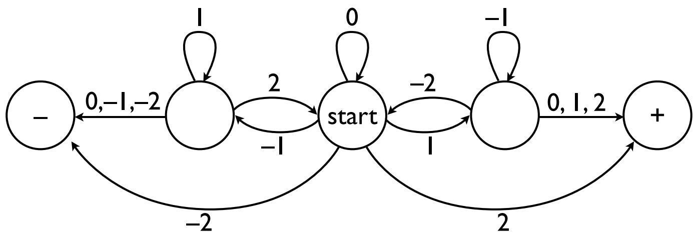

Fig. 4.1: Finite automaton for the proof of Lemma 4.1.

start state and two additional states). We interpret the output of the automaton to be $+1$ and -1 at the two terminal states, respectively, and 0 otherwise. A string $z = (z_n, z_{n-1}, \dots, z_1, 0) \in \{0, \pm 1, \pm 2\}^{n+1}$, when read by the automaton left to right, forces it to output exactly $\operatorname{sgn}(\sum_{i=1}^n 2^i z_i)$. If the automaton is currently at a nonterminal state, that state is determined uniquely by the last two symbols read. Thus, the output of the automaton on input $z = (z_n, z_{n-1}, \dots, z_1, 0) \in \{0, \pm 1, \pm 2\}^{n+1}$ is

$$\operatorname{sgn}\left(\sum_{i=0}^{n} 2^i \alpha(z_{i+2}, z_{i+1}, z_i)\right)$$

for a suitable map $\alpha\colon \{0, \pm 1, \pm 2\}^3 \to \{0, -1, +1\}$, where we adopt the shorthand $z_{n+1} = z_{n+2} = z_0 = 0$. Put

$$A_M(z) = \frac{1 + \sum_{i=0}^n M^{i+1} \alpha(z_{i+2}, z_{i+2}, z_i)}{1 + \sum_{i=0}^n M^{i+1} |\alpha(z_{i+2}, z_{i+2}, z_i)|}.$$

By interpolation, α and $|\alpha|$ are polynomials of degree at most $4 \times 4 \times 4 = 64$, so that the numerator and denominator of A_M have degree at most 64. On the other hand, we have $\|f - A_M\|_\infty \to 0$ as $M \to \infty$. ☐

We are now prepared to prove our desired upper bounds for halfspaces.

Theorem 4.2. *Let $f\colon \{-1, +1\}^{nk} \to \{-1, +1\}$ be the function given by*

$$f(x) = \operatorname{sgn}\left(1 + \sum_{i=1}^{n} \sum_{j=1}^{k} 2^i x_{ij}\right). \tag{4.1}$$

Then

$$R^+(f, 64k\lceil \log k \rceil + 1) = 0. \tag{4.2}$$

In addition, for all integers $d \geqslant 1$,

$$R^+(f, d) \leqslant 1 - (k2^{n+1})^{-1/d}. \tag{4.3}$$

Proof. Theorem 2.4 immediately implies (4.3) in view of the representation (4.1). It remains to prove (4.2). In the degenerate case $k = 1$, we have $f \equiv x_{n1}$ and thus (4.2) holds. In what follows, we assume that $k \geqslant 2$ and put $\Delta = \lceil \log k \rceil$. We adopt the convention that $x_{ij} \equiv 0$ for $i > n$. For $\ell = 0, 1, 2, \dots$, define

$$S_\ell = \sum_{i=1}^{\Delta} \sum_{j=1}^{k} 2^{i-1} x_{\ell\Delta + i, j}.$$

Then

$$\sum_{i=1}^{n}\sum_{j=1}^{k} 2^{i-1}x_{ij} = \left(S_0 + 2^{2\Delta}S_2 + 2^{4\Delta}S_4 + 2^{6\Delta}S_6 + \cdots\right) + \left(2^{\Delta}S_1 + 2^{3\Delta}S_3 + 2^{5\Delta}S_5 + 2^{7\Delta}S_7 + \cdots\right). \quad (4.4)$$

Now, each S_ℓ is an integer in $[-2^{2\Delta}+1, 2^{2\Delta}-1]$ and therefore admits a representation as

$$S_\ell = z_{\ell,1} + 2z_{\ell,2} + 2^2 z_{\ell,3} + \cdots + 2^{2\Delta-1}z_{\ell,2\Delta},$$

where $z_{\ell,1},\dots,z_{\ell,2\Delta} \in \{-1,0,+1\}$. Furthermore, each S_ℓ only depends on $k\Delta$ of the original variables x_{ij}, whence $z_{\ell,1},\dots,z_{\ell,2\Delta}$ can all be viewed as polynomials of degree at most $k\Delta$ in the original variables. Rewriting (4.4),

$$\sum_{i=1}^{n}\sum_{j=1}^{k} 2^{i-1}x_{ij} = \left(\sum_{i\geqslant 1} 2^{i-1} z_{\ell(i),j(i)}\right) + \left(\sum_{i\geqslant \Delta+1} 2^{i-1} z_{\ell'(i),j'(i)}\right)$$

for appropriate indexing functions $\ell(i), \ell'(i), j(i), j'(i)$. Thus,

$$f(x) \equiv \operatorname{sgn}\left(1 + \sum_{i=1}^{\Delta} 2^i \underbrace{z_{\ell(i),j(i)}} + \sum_{i\geqslant \Delta+1} 2^i \underbrace{\left(z_{\ell(i),j(i)} + z_{\ell'(i),j'(i)}\right)}\right).$$

Since the underbraced expressions range in $\{0,\pm1,\pm2\}$ and are polynomials of degree at most $k\Delta$ in the original variables, Lemma 4.1 implies (4.2). ∎

Theorem 4.2 settles all upper bounds on $\operatorname{rdeg}_\epsilon(f)$ in Theorem 1.6.

4.2. Preparatory work. This section sets the stage for the rational approximation lower bounds, with some preparatory results about halfspaces. It will be convenient to establish some additional notation, for use in this section only. Here, we typeset real vectors in boldface $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{z}, \mathbf{v})$ to better distinguish them from scalars. The ith component of a vector $\mathbf{x} \in \mathbb{R}^n$ is denoted $(\mathbf{x})_i$, while the symbol $\mathbf{x}_i$ is reserved for another *vector* from some enumeration. In keeping with this convention, we let $\mathbf{e}_i$ denote the vector with 1 in the ith component and zeroes everywhere else. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the vector $\mathbf{xy} \in \mathbb{R}^n$ is given by $(\mathbf{xy})_i \equiv (\mathbf{x})_i(\mathbf{y})_i$. This convention gives, among other things, $\mathbf{x}^k = \mathbf{xx}\cdots\mathbf{x}$ (k times). More generally, for a polynomial p on $\mathbb{R}^k$ and vectors $\mathbf{x}_1,\dots,\mathbf{x}_k \in \mathbb{R}^n$, we define $p(\mathbf{x}_1,\dots,\mathbf{x}_k) \in \mathbb{R}^n$ by $(p(\mathbf{x}_1,\dots,\mathbf{x}_k))_i = p((\mathbf{x}_1)_i,\dots,(\mathbf{x}_k)_i)$. The expectation of a random variable $\mathbf{x} \in \mathbb{R}^n$ is defined componentwise, i.e., the vector $\mathbf{E}[\mathbf{x}] \in \mathbb{R}^n$ is given by $(\mathbf{E}[\mathbf{x}])_i \equiv \mathbf{E}[(\mathbf{x})_i]$.

For convenience, we adopt the notational shorthand $\alpha^0 = 1$ for all $\alpha \in \mathbb{R}$. In particular, if $\mathbf{x} \in \mathbb{R}^n$ is a given vector, then $\mathbf{x}^0 = (1,1,\dots,1) \in \mathbb{R}^n$. A scalar $\alpha \in \mathbb{R}$, when interpreted as a vector, stands for $(\alpha,\alpha,\dots,\alpha)$. This shorthand allows one to speak of $\operatorname{span}\{1,\mathbf{z},\mathbf{z}^2,\dots,\mathbf{z}^k\}$, for example, where $\mathbf{z} \in \mathbb{R}^n$ is a given vector.

Theorem 4.3. *Let N and m be positive integers. Then reals $\alpha_0,\alpha_1,\dots,\alpha_{4m}$ exist with the following property: for each $\mathbf{b} \in \{0,1\}^N$, there is a probability distribution $\mu_{\mathbf{b}}$ on $\{0,\pm1,\dots,\pm m\}^N$ such that*

$$\mathop{\mathbf{E}}_{\mathbf{v}\sim\mu_{\mathbf{b}}}[(2\mathbf{v}+\mathbf{b})^d] = (\alpha_d,\alpha_d,\dots,\alpha_d), \qquad d = 0,1,2,\dots,4m.$$

Proof. Let λ_0 and λ_1 be the distributions on $\{0, \pm 1, \ldots, \pm m\}$ given by

$$\lambda_0(t) = 16^{-m}\binom{4m+1}{2m+2t}, \qquad \lambda_1(t) = 16^{-m}\binom{4m+1}{2m+2t+1}.$$

Then for $d = 0, 1, \ldots, 4m$, Fact 2.1 gives

$$\mathop{\mathbf{E}}_{t\sim\lambda_0}[(2t)^d] - \mathop{\mathbf{E}}_{t\sim\lambda_1}[(2t+1)^d] = 16^{-m}\sum_{t=0}^{4m+1}(-1)^t\binom{4m+1}{t}(t-2m)^d = 0. \tag{4.5}$$

Now, let $\mu_{\mathbf{b}} = \lambda_{(\mathbf{b})_1} \times \lambda_{(\mathbf{b})_2} \times \cdots \times \lambda_{(\mathbf{b})_N}$. Then in view of (4.5), the theorem holds by letting $\alpha_d = \mathbf{E}_{\lambda_0}[(2t)^d]$ for $d = 0, 1, 2, \ldots, 4m$. ☐

Using the previous theorem, we will now establish another auxiliary result pertaining to halfspaces.

THEOREM 4.4. *Put* $\mathbf{z} = (-2^n, -2^{n-1}, \ldots, -2^0, 2^0, \ldots, 2^{n-1}, 2^n) \in \mathbb{R}^{2n+2}$. *There are random variables* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1} \in \{0, \pm 1, \pm 2, \ldots, \pm(3n+1)\}^{2n+2}$ *such that:*

$$\sum_{i=1}^{n+1} 2^{i-1}\mathbf{x}_i \equiv \mathbf{z} \tag{4.6}$$

and

$$\mathbf{E}\left[\prod_{i=1}^{n}\mathbf{x}_i^{d_i}\right] \in \operatorname{span}\{(1, 1, \ldots, 1)\} \tag{4.7}$$

for $d_1, \ldots, d_n \in \{0, 1, \ldots, 4n\}$.

Proof. Let

$$\mathbf{x}_i = 2\mathbf{y}_i - \mathbf{y}_{i-1} + \mathbf{e}_{n+1+i} - \mathbf{e}_{n+2-i}, \qquad i = 1, 2, \ldots, n+1,$$

where $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{n+1}$ are suitable random variables with $\mathbf{y}_0 \equiv \mathbf{y}_{n+1} \equiv 0$. Then property (4.6) is immediate. We will construct $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{n+1}$ such that the remaining property (4.7) holds as well.

Let $N = 2n+2$ and $m = n$ in Theorem 4.3. Then reals $\alpha_0, \alpha_1, \ldots, \alpha_{4n}$ exist with the property that for each $\mathbf{b} \in \{0,1\}^{2n+2}$, a probability distribution $\mu_{\mathbf{b}}$ can be found on $\{0, \pm 1, \ldots, \pm n\}^{2n+2}$ such that

$$\mathop{\mathbf{E}}_{\mathbf{v}\sim\mu_{\mathbf{b}}}[(2\mathbf{v}+\mathbf{b})^d] = \alpha_d(1, 1, \ldots, 1), \qquad d = 0, 1, \ldots, 4n. \tag{4.8}$$

Now, we will specify the distribution of $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n$ by giving an algorithm for generating $\mathbf{y}_i$ from $\mathbf{y}_{i-1}$. First, recall that $\mathbf{y}_0 \equiv \mathbf{y}_{n+1} \equiv 0$. The algorithm for generating $\mathbf{y}_i$ given $\mathbf{y}_{i-1}$ ($i = 1, 2, \ldots, n$) is as follows.

(1) Let $\mathbf{u}$ be the unique integer vector such that $2\mathbf{u} - \mathbf{y}_{i-1} + \mathbf{e}_{n+1+i} - \mathbf{e}_{n+2-i} \in \{0,1\}^{2n+2}$. In other words, $(\mathbf{u})_j = \lceil (\mathbf{y}_{i-1} - \mathbf{e}_{n+1+i} + \mathbf{e}_{n+2-i})_j/2 \rceil$ for each component $j = 1, 2, \ldots, 2n+2$.

(2) Draw a random vector $\mathbf{v} \sim \mu_{\mathbf{b}}$, where $\mathbf{b} = 2\mathbf{u} - \mathbf{y}_{i-1} + \mathbf{e}_{n+1+i} - \mathbf{e}_{n+2-i}$.

(3) Set $\mathbf{y}_i = \mathbf{v} + \mathbf{u}$.

One easily verifies that $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{n+1} \in \{0, \pm 1, \ldots, \pm 3n\}^{2n+2}$.

Let R denote the resulting joint distribution of $(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{n+1})$. Let $i \leqslant n$. Then conditioned on any fixed value of $(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{i-1})$ in the support of R, the random variable $\mathbf{x}_i$ is by definition independent of $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ and is distributed identically to $2\mathbf{v} + \mathbf{b}$, for some fixed vector $\mathbf{b} \in \{0,1\}^{2n+2}$ and a random variable $\mathbf{v} \sim \mu_{\mathbf{b}}$. In view of (4.8), we conclude that

$$\mathbf{E}\left[\prod_{i=1}^{n} \mathbf{x}_i^{d_i}\right] = (1,1,\ldots,1)\prod_{i=1}^{n} \alpha_{d_i}$$

for all $d_1, d_2, \ldots, d_n \in \{0,1,\ldots,4n\}$, which establishes (4.7). It remains to note that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \{-2n, -2n+1, \ldots, -1, 0, 1, \ldots, 2n, 2n+1\}^{2n+2}$, whereas $\mathbf{x}_{n+1} = -\mathbf{y}_n + \mathbf{e}_{2n+2} - \mathbf{e}_1 \in \{0, \pm 1, \ldots, \pm(3n+1)\}^{2n+2}$. ∎

At last, we arrive at the main theorem of this section, which will play a crucial role in our analysis of the rational approximation of halfspaces.

THEOREM 4.5. *For $i = 0, 1, 2, \ldots, n$, define*

$$A_i = \left\{ (x_1, \ldots, x_{n+1}) \in \{0, \pm 1, \ldots, \pm(3n+1)\}^{n+1} : \quad \sum_{j=1}^{n+1} 2^{j-1} x_j = 2^i \right\}.$$

Let $p(x_1, \ldots, x_{n+1})$ be a real polynomial with sign $(-1)^i$ on A_i $(i = 0,1,2,\ldots,n)$ and sign $(-1)^{i+1}$ on $-A_i$ $(i = 0,1,2,\ldots,n)$. Then

$$\deg p \geqslant 2n+1.$$

Proof. For the sake of contradiction, suppose that p has degree no greater than $2n$. Put $\mathbf{z} = (-2^n, -2^{n-1}, \ldots, -2^0, 2^0, \ldots, 2^{n-1}, 2^n)$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$ be the random variables constructed in Theorem 4.4. By (4.7) and the identity $\mathbf{x}_{n+1} \equiv 2^{-n}\mathbf{z} - \sum_{i=1}^{n} 2^{i-n-1}\mathbf{x}_i$, we have

$$\mathbf{E}[p(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})] \in \operatorname{span}\{1, \mathbf{z}, \mathbf{z}^2, \ldots, \mathbf{z}^{2n}\},$$

whence $\mathbf{E}[p(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})] = q(\mathbf{z})$ for a univariate polynomial $q \in P_{2n}$. In view of (4.6) and the assumed sign behavior of p, we have $\operatorname{sgn} q(2^i) = (-1)^i$ and $\operatorname{sgn} q(-2^i) = (-1)^{i+1}$, for $i = 0,1,2,\ldots,n$. Therefore, q has at least $2n+1$ roots. Since $q \in P_{2n}$, we arrive at a contradiction. Hence, the assumed polynomial p does not exist. ∎

REMARK. The passage $p \mapsto q$ in the proof of Theorem 4.5 is precisely the degree-nonincreasing linear map $M\colon \mathbb{R}[x_1, x_2, \ldots, x_{n+1}] \to \mathbb{R}[x]$ described previously in the Introduction.

4.3. Lower bounds. The purpose of this section is to prove that the canonical halfspace cannot be approximated well by a rational function of low degree. A starting point in our discussion is a criterion for inapproximability by low-degree rational functions, which is applicable not only to halfspaces but any odd Boolean functions on Euclidean space.

THEOREM 4.6 (Criterion for inapproximability). *Fix a nonempty finite subset $S \subset \mathbb{R}^m$ with $S \cap -S = \varnothing$. Define $f\colon S \cup -S \to \{-1, +1\}$ by*

$$f(x) = \begin{cases} +1, & x \in S, \\ -1, & x \in -S. \end{cases}$$

Let ψ be a real function such that

$$\psi(x) > \delta|\psi(-x)|, \qquad x \in S, \tag{4.9}$$

for some $\delta \in (0,1)$ *and*

$$\sum_{S\cup -S} \psi(x)u(x) = 0 \tag{4.10}$$

for every polynomial u *of degree at most* d*. Then*

$$R^+(f,d) \geqslant \frac{2\delta}{1+\delta}.$$

Proof. Fix polynomials p, q of degree at most d such that q is positive on $S \cup -S$. Put

$$\epsilon = \max_{S\cup -S} \left| f(x) - \frac{p(x)}{q(x)} \right|.$$

We assume that $\epsilon < 1$ since otherwise there is nothing to show. For $x \in S$,

$$(1-\epsilon)q(x) \leqslant p(x) \leqslant (1+\epsilon)q(x) \tag{4.11}$$

and

$$(1-\epsilon)q(-x) \leqslant -p(-x) \leqslant (1+\epsilon)q(-x). \tag{4.12}$$

Consider the polynomial $u(x) = q(x) + q(-x) + p(x) - p(-x)$. Equations (4.11) and (4.12) show that for $x \in S$, one has $u(x) \geqslant (2-\epsilon)\{q(x) + q(-x)\}$ and $|u(-x)| \leqslant \epsilon\{q(x) + q(-x)\}$, whence

$$u(x) \geqslant \left(\frac{2}{\epsilon} - 1\right) |u(-x)|, \qquad x \in S. \tag{4.13}$$

We also note that

$$u(x) > 0, \qquad x \in S. \tag{4.14}$$

Since u has degree at most d, we have by (4.10) that

$$\sum_{x\in S} \{\psi(x)u(x) + \psi(-x)u(-x)\} = \sum_{S\cup -S} \psi(x)u(x) = 0,$$

whence

$$\psi(x)u(x) \leqslant |\psi(-x)u(-x)|$$

for some $x \in S$. At the same time, it follows from (4.9), (4.13), and (4.14) that

$$\psi(x)u(x) > \delta\left(\frac{2}{\epsilon} - 1\right)|\psi(-x)u(-x)|, \qquad x \in S.$$

We immediately obtain $\delta(\{2/\epsilon\} - 1) < 1$, as was to be shown. ∎

Remark. The method of Theorem 4.6 amounts to reformulating (4.13) and (4.14) as a linear program and exhibiting a solution to its dual. The presentation above does not explicitly use the language of linear programs or appeal to duality, however, because our goal is solely to prove the correctness of the method and not its completeness.

Using the criterion of Theorem 4.6 and the preparatory work in Section 4.2, we now establish a key lower bound for the rational approximation of halfspaces within constant error.

THEOREM 4.7. *Let* $f\colon \{0, \pm 1, \ldots, \pm(3n+1)\}^{n+1} \to \{-1, +1\}$ *be given by*

$$f(x) = \operatorname{sgn}\left(1 + \sum_{i=1}^{n+1} 2^i x_i\right).$$

Then

$$R^+(f, n) = \Omega(1).$$

Proof. Let $A_0, A_1, \ldots, A_n$ be as defined in Theorem 4.5. Put $A = \bigcup A_i$ and define $g\colon A \cup -A \to \{-1, +1\}$ by

$$g(x) = \begin{cases} (-1)^i, & x \in A_i, \\ (-1)^{i+1}, & x \in -A_i. \end{cases}$$

Then $\deg_{\pm}(g) > 2n$ by Theorem 4.5. As a result, Theorem 2.2 guarantees the existence of a function $\phi\colon A \cup -A \to \mathbb{R}$, not identically zero, such that

$$\phi(x) g(x) \geqslant 0, \qquad x \in A \cup -A, \tag{4.15}$$

and

$$\sum_{A \cup -A} \phi(x) u(x) = 0 \tag{4.16}$$

for every polynomial u of degree at most $2n$. Put

$$p(x) = \prod_{j=0}^{n-1} \left(-2^j\sqrt{2} + \sum_{i=1}^{n+1} 2^{i-1} x_i\right)$$

and

$$\psi(x) = (-1)^n \{\phi(x) - \phi(-x)\} p(x).$$

Clearly, p does not vanish on $A \cup -A$. Define $S = A \setminus \psi^{-1}(0)$. Then $S \neq \varnothing$ by (4.15) and the fact that ϕ is not identically zero on $A \cup -A$. For $x \in S$, we have $\psi(-x) \neq 0$ and

$$\begin{aligned} \frac{|\psi(x)|}{|\psi(-x)|} = \frac{|p(x)|}{|p(-x)|} &\geqslant \min_{i \in \mathbb{Z}} \left\{ \prod_{j=0}^{n-1} \frac{|-2^j\sqrt{2} + 2^i|}{2^j\sqrt{2} + 2^i} \right\} > \min_{i \in \mathbb{Z}} \left\{ \prod_{j=-\infty}^{\infty} \frac{|-2^j\sqrt{2} + 2^i|}{2^j\sqrt{2} + 2^i} \right\} \\ &= \prod_{j=-\infty}^{\infty} \frac{|2^j\sqrt{2} - 1|}{2^j\sqrt{2} + 1} > \left(\prod_{j=1}^{\infty} \frac{2^{j/2} - 1}{2^{j/2} + 1} \right)^2 > \exp(-9\sqrt{2}), \end{aligned} \tag{4.17}$$

where the final step uses the bound $(a-1)/(a+1) > \exp(-2.5/a)$, valid for $a \geqslant \sqrt{2}$.

It follows from (4.15) that $\operatorname{sgn}\{\phi(x) - \phi(-x)\} = g(x)$ for all $x \in S$. Since in addition $\operatorname{sgn} p(x) = (-1)^n g(x)$ for all $x \in A \supseteq S$, we conclude that ψ is positive on S. Now (4.17) gives

$$\psi(x) > \exp(-9\sqrt{2})\, |\psi(-x)|, \qquad x \in S. \tag{4.18}$$

For any polynomial u of degree no greater than n, we infer from (4.16) that

$$\sum_{S \cup -S} \psi(x)u(x) = (-1)^n \sum_{A \cup -A} \{\phi(x) - \phi(-x)\}u(x)p(x) = 0. \tag{4.19}$$

Since f is positive on S and negative on $-S$, the proof is now complete in view of (4.18), (4.19), and Theorem 4.6. ☐

We have reached the main result of this section, which extends Theorem 4.7 to any subconstant approximation error and to halfspaces on the hypercube. This settles the lower bounds in Theorem 1.6 from the Introduction.

THEOREM 4.8. *Let* $F\colon \{-1,+1\}^{m^2} \to \{-1,+1\}$ *be given by*

$$F(x) = \operatorname{sgn}\left(1 + \sum_{i=1}^{m}\sum_{j=1}^{m} 2^i x_{ij}\right).$$

Then for $d < m/14$,

$$R(F, d) \geqslant 1 - 2^{-\Theta(m/d)}. \tag{4.20}$$

Proof. We may assume that $m \geqslant 14$, the claim being trivial otherwise. Consider the function $G\colon \{-1,+1\}^{(n+1)(6n+2)} \to \{-1,+1\}$ given by

$$G(x) = \operatorname{sgn}\left(1 + \sum_{i=1}^{n+1}\sum_{j=1}^{6n+2} 2^i x_{ij}\right),$$

where $n = \lfloor (m-2)/6 \rfloor$. For every $\epsilon > R^+(G, n)$, Proposition 2.7 provides a rational function A on $\mathbb{R}^{n+1}$ of degree at most n such that, on the domain of G,

$$\left| G(x) - A\left(\ldots, \sum_{j=1}^{6n+2} x_{ij}, \ldots\right)\right| < \epsilon$$

and the denominator of A is positive. Letting f be the function in Theorem 4.7, it follows that $|f(x_1, \ldots, x_{n+1}) - A(2x_1, \ldots, 2x_{n+1})| < \epsilon$ on the domain of f, whence

$$R^+(G, n) = \Omega(1). \tag{4.21}$$

We now claim that either $G(x)$ or $-G(-x)$ is a subfunction of F. For example, consider the following substitution for the variables x_{ij} for which $i > n+1$ or $j > 6n+2$:

$$\begin{aligned}
x_{mj} &\leftarrow (-1)^j, && (1 \leqslant j \leqslant m),\\
x_{ij} &\leftarrow (-1)^{j+1}, && (n+1 < i < m, \quad 1 \leqslant j \leqslant m),\\
x_{ij} &\leftarrow (-1)^{j+1}, && (1 \leqslant i \leqslant n+1, \quad j > 6n+2).
\end{aligned}$$

After this substitution, F is a function of the remaining variables x_{ij} and is equivalent to $G(x)$ if m is even, and to $-G(-x)$ if m is odd. In either case, (4.21) implies that

$$R^+(F,n) = \Omega(1). \tag{4.22}$$

Theorem 2.5 shows that

$$R(F,n/2) \leqslant 1 - \left(\frac{1-R(F,d)}{2}\right)^{1/\lfloor n/(2d)\rfloor}$$

for $d = 1, 2, \dots, \lfloor n/2 \rfloor$, which yields (4.20) in light of (2.2) and (4.22). $\square$

5. Rational approximation of the majority function. The goal of this section is to determine $R^+(\mathrm{MAJ}_n, d)$ for each integer d, i.e., to determine the least error to which a degree-d multivariate rational function can approximate the majority function. As is frequently the case with symmetric Boolean functions, this multivariate problem is equivalent to a univariate question. Specifically, given an integer d and a finite set $S \subset \mathbb{R}$, we define

$$R^+(d,S) \;=\; \inf_{p,q} \max_{t\in S} \left|\operatorname{sgn} t - \frac{p(t)}{q(t)}\right|,$$

where the infimum ranges over $p, q \in P_d$ such that q is positive on S. In other words, we study how well a rational function of a given degree can approximate the sign function over a finite support. We prove:

THEOREM 5.1 (Rational approximation of MAJORITY). *Let n, d be positive integers. Abbreviate $R = R^+(d, \{\pm 1, \pm 2, \dots, \pm n\})$. For $1 \leqslant d \leqslant \log n$,*

$$\exp\left\{-\Theta\left(\frac{1}{n^{1/(2d)}}\right)\right\} \leqslant R < \exp\left\{-\frac{1}{n^{1/d}}\right\}.$$

For $\log n < d < n$,

$$R = \exp\left\{-\Theta\left(\frac{d}{\log(2n/d)}\right)\right\}.$$

For $d \geqslant n$,

$$R = 0.$$

Moreover, the rational approximant is constructed explicitly in each case.

Theorem 5.1 is the main result of this section. We establish it in the next two subsections, giving separate treatment to the cases $d \leqslant \log n$ and $d > \log n$ (see Theorems 5.3 and 5.8, respectively). In the concluding subsection, we give the promised proof that $R^+(d, \{\pm 1, \dots, \pm n\})$ and $R^+(\mathrm{MAJ}_n, d)$ are essentially equivalent.

5.1. Low-degree approximation. We start by specializing the criterion of Theorem 4.6 to the problem of approximating the sign function on $\{\pm 1, \pm 2, \dots, \pm n\}$.

THEOREM 5.2. *Let d be an integer, $0 \leqslant d \leqslant 2n-1$. Fix a nonempty subset $S \subseteq \{1, 2, \dots, n\}$. Suppose that there exists a real $\delta \in (0,1)$ and a polynomial $r \in P_{2n-d-1}$ that vanishes on $\{-n, \dots, n\} \setminus (S \cup -S)$ and obeys*

$$(-1)^t r(t) > \delta |r(-t)|, \qquad t \in S. \tag{5.1}$$

Then

$$R^+(d, S \cup -S) \geqslant \frac{2\delta}{1+\delta}. \tag{5.2}$$

Proof. Define $f\colon S \cup -S \to \{-1,+1\}$ by $f(t) = \operatorname{sgn} t$. Define $\psi\colon S \cup -S \to \mathbb{R}$ by $\psi(t) = (-1)^t \binom{2n}{n+t} r(t)$. Then (5.1) takes on the form

$$\psi(t) > \delta|\psi(-t)|, \qquad t \in S. \tag{5.3}$$

For every polynomial u of degree at most d, we have

$$\sum_{S\cup -S} \psi(t)u(t) = \sum_{t=-n}^{n} (-1)^t \binom{2n}{n+t} r(t)u(t) = 0 \tag{5.4}$$

by Fact 2.1. Now (5.2) is immediate from (5.3), (5.4), and Theorem 4.6. ∎

Using Theorem 5.2, we will now determine the optimal error in the approximation of the majority function by rational functions of degree up to $\log n$.

Theorem 5.3 (Low-degree rational approximation of majority). *Let d be an integer, $1 \leqslant d \leqslant \log n$. Then*

$$\exp\left\{-\Theta\left(\frac{1}{n^{1/(2d)}}\right)\right\} \leqslant R^+(d, \{\pm 1, \pm 2, \dots, \pm n\}) < \exp\left\{-\frac{1}{n^{1/d}}\right\}.$$

Proof. The upper bound is immediate from Newman's Theorem 2.4. For the lower bound, put $\Delta = \lfloor n^{1/d} \rfloor \geqslant 2$ and $S = \{1, \Delta, \Delta^2, \dots, \Delta^d\}$. Define $r \in P_{2n-d-1}$ by

$$r(t) = (-1)^n \prod_{i=0}^{d-1} (t - \Delta^i \sqrt{\Delta}) \prod_{i\in\{-n,\dots,n\}\setminus(S\cup -S)} (t-i).$$

For $j = 0, 1, 2, \dots, d$,

$$\begin{aligned}\frac{|r(\Delta^j)|}{|r(-\Delta^j)|} &= \prod_{i=0}^{j-1} \frac{\Delta^j - \Delta^i\sqrt{\Delta}}{\Delta^j + \Delta^i\sqrt{\Delta}} \prod_{i=j}^{d-1} \frac{\Delta^i\sqrt{\Delta} - \Delta^j}{\Delta^i\sqrt{\Delta} + \Delta^j} > \left(\prod_{i=1}^{\infty} \frac{\Delta^{i/2}-1}{\Delta^{i/2}+1}\right)^2 \\ &> \exp\left\{-5\sum_{i=1}^{\infty} \frac{1}{\Delta^{i/2}}\right\} > \exp\left\{-\frac{18}{\sqrt{\Delta}}\right\},\end{aligned}$$

where we used the bound $(a-1)/(a+1) > \exp(-2.5/a)$, valid for $a \geqslant \sqrt{2}$. Since $\operatorname{sgn} r(t) = (-1)^t$ for $t \in S$, we conclude that

$$(-1)^t r(t) > \exp\left\{-\frac{18}{\sqrt{\Delta}}\right\} |r(-t)|, \qquad t \in S.$$

Since in addition r vanishes on $\{-n, \dots, n\} \setminus (S \cup -S)$, we infer from Theorem 5.2 that $R^+(d, S \cup -S) \geqslant \exp\{-18/\sqrt{\Delta}\}$. ∎

5.2. High-degree approximation. To start with, we need to accurately estimate products of the form $\prod_i (\Delta^i+1)/(\Delta^i-1)$ for all $\Delta > 1$. A suitable *lower* bound was already given by Newman [32, Lem. 1]:

Lemma 5.4 (Newman). *For all* $\Delta > 1$,

$$\prod_{i=1}^{n} \frac{\Delta^i+1}{\Delta^i-1} > \exp\left\{\frac{2(\Delta^n-1)}{\Delta^n(\Delta-1)}\right\}.$$

Proof. Immediate from $(a+1)/(a-1) > \exp(2/a)$, which is valid for $a > 1$. ◻

We will need a corresponding upper bound:

Lemma 5.5. *For all* $\Delta > 1$,

$$\prod_{i=1}^{\infty} \frac{\Delta^i+1}{\Delta^i-1} < \exp\left\{\frac{4}{\Delta-1}\right\}.$$

Proof. Let $k \geqslant 0$ be an integer. By the binomial theorem, $\Delta^i \geqslant (\Delta-1)i+1$ for integers $i \geqslant 0$. As a result,

$$\prod_{i=1}^{k} \frac{\Delta^i+1}{\Delta^i-1} \leqslant \prod_{i=1}^{k} \frac{1}{i}\left(i+\frac{2}{\Delta-1}\right) \leqslant \binom{k+\left\lceil\frac{2}{\Delta-1}\right\rceil}{k}.$$

Also,

$$\prod_{i=k+1}^{\infty} \frac{\Delta^i+1}{\Delta^i-1} < \prod_{i=0}^{\infty}\left(1+\frac{2}{(\Delta^{k+1}-1)\Delta^i}\right) < \exp\left\{\frac{2\Delta}{(\Delta^{k+1}-1)(\Delta-1)}\right\}.$$

Setting $k = k(\Delta) = \left\lfloor\frac{2}{\Delta-1}\right\rfloor$, we conclude that

$$\prod_{i=1}^{\infty} \frac{\Delta^i+1}{\Delta^i-1} < \exp\left\{\frac{C}{\Delta-1}\right\},$$

where

$$C = \sup_{\Delta>1}\left\{(\Delta-1)\ln\binom{k(\Delta)+\left\lceil\frac{2}{\Delta-1}\right\rceil}{k(\Delta)} + \frac{2\Delta}{\Delta^{k(\Delta)+1}-1}\right\} < 4. \qquad ◻$$

We will also need the following binomial estimate.

Lemma 5.6. *Put* $p(t) = \prod_{i=1}^{n}\left(t-i-\frac{1}{2}\right)$. *Then*

$$\max_{t=1,2,\dots,n+1}\left|\frac{p(-t)}{p(t)}\right| \leqslant \Theta(16^n).$$

Proof. For $t = 1, 2, \dots, n+1$, we have

$$|p(t)| = \frac{(2t-2)!(2n-2t+2)!}{4^n(t-1)!(n-t+1)!}, \qquad |p(-t)| = \frac{t!(2n+2t+1)!}{4^n(2t+1)!(n+t)!}.$$

As a result,

$$\left|\frac{p(-t)}{p(t)}\right| = \frac{t}{2t+1} \cdot \frac{\binom{2n+2t+1}{2t}\binom{2n+1}{n+t}}{\binom{2t-2}{t-1}\binom{2n-2t+2}{n-t+1}} \leqslant \frac{\Theta\left(\frac{2^{4n}}{\sqrt{n}}\right)\Theta\left(\frac{2^{2n}}{\sqrt{n}}\right)}{\Theta\left(\frac{2^{2n}}{n}\right)},$$

which gives the sought bound. ∎

Our construction requires one additional ingredient.

LEMMA 5.7. *Let n, d be integers, $1 \leqslant d \leqslant n/55$. Consider the polynomial $p(t) = \prod_{i=1}^{d-1}(t - d\Delta^i\sqrt{\Delta})$, where $\Delta = (n/d)^{1/d}$. Then*

$$\min_{j=1,\dots,d}\left|\frac{p(\lfloor d\Delta^j\rfloor)}{p(-\lfloor d\Delta^j\rfloor)}\right| > \exp\left\{-\frac{4\ln 3d}{\ln(n/d)} - \frac{8}{\sqrt{\Delta}-1}\right\}.$$

Proof. Fix $j = 1, 2, \dots, d$. Then for each $i = 1, 2, \dots, j-1$,

$$d\Delta^j - d\Delta^i\sqrt{\Delta} \geqslant d\left(\Delta^{j-i-\frac{1}{2}} - 1\right) \geqslant d\left(j - i - \frac{1}{2}\right)\ln\Delta \geqslant \frac{1}{2}(j-i)\ln\frac{n}{d},$$

where the second step uses the inequality $a - 1 \geqslant \ln a$, valid for all positive a. Hence,

$$\begin{aligned}\prod_{i=1}^{j-1}\left(1 - \frac{1}{d\Delta^j - d\Delta^i\sqrt{\Delta}}\right) &\geqslant \exp\left\{-\frac{4}{\ln(n/d)}\sum_{i=1}^{j-1}\frac{1}{j-i}\right\}\\ &\geqslant \exp\left\{-\frac{4\ln 3d}{\ln(n/d)}\right\}.\end{aligned} \tag{5.5}$$

For brevity, let ξ stand for the final expression in (5.5). Since $1 \leqslant d \leqslant n/55$, we have $\lfloor d\Delta^j\rfloor - d\Delta^{j-1}\sqrt{\Delta} > 1$. As a result,

$$\begin{aligned}\left|\frac{p(\lfloor d\Delta^j\rfloor)}{p(-\lfloor d\Delta^j\rfloor)}\right| &\geqslant \prod_{i=1}^{j-1}\frac{d\Delta^j - 1 - d\Delta^i\sqrt{\Delta}}{d\Delta^j + d\Delta^i\sqrt{\Delta}}\prod_{i=j}^{d-1}\frac{d\Delta^i\sqrt{\Delta} - d\Delta^j}{d\Delta^i\sqrt{\Delta} + d\Delta^j}\\ &\geqslant \xi\prod_{i=1}^{j-1}\frac{d\Delta^j - d\Delta^i\sqrt{\Delta}}{d\Delta^j + d\Delta^i\sqrt{\Delta}}\prod_{i=j}^{d-1}\frac{d\Delta^i\sqrt{\Delta} - d\Delta^j}{d\Delta^i\sqrt{\Delta} + d\Delta^j} \qquad \text{by (5.5)}\\ &> \xi\left(\prod_{i=1}^{\infty}\frac{\Delta^{i/2}-1}{\Delta^{i/2}+1}\right)^2\\ &\geqslant \xi\exp\left\{-\frac{8}{\sqrt{\Delta}-1}\right\},\end{aligned}$$

where the last inequality holds by Lemma 5.5. ∎

We have reached the main result of this subsection.

THEOREM 5.8 (High-degree rational approximation of MAJORITY). *Let d be an integer, $\log n < d \leqslant n-1$. Then*

$$R^+(d, \{\pm1, \pm2, \dots, \pm n\}) = \exp\left\{-\Theta\left(\frac{d}{\log(2n/d)}\right)\right\}.$$

Also,

$$R^+(n, \{\pm 1, \pm 2, \ldots, \pm n\}) = 0.$$

Proof. The final statement in the theorem follows at once by considering the rational function $\{p(t) - p(-t)\}/\{p(t) + p(-t)\}$, where $p(t) = \prod_{i=1}^{n}(t+i)$.

Now assume that $\log n < d < n/55$. Let

$$k = \left\lceil \frac{d}{\log(n/d)} \right\rceil, \qquad \Delta = \left(\frac{n}{d}\right)^{1/d}.$$

Define sets

$$\begin{aligned} S_1 &= \{1, 2, \ldots, k\}, \\ S_2 &= \{\lfloor d\Delta^i \rfloor : i = 1, 2, \ldots, d\}, \\ S \ &= S_1 \cup S_2. \end{aligned}$$

Consider the polynomial

$$r(t) = (-1)^n r_1(t) r_2(t) \prod_{i \in \{-n,\ldots,n\} \setminus (S \cup -S)} (t - i),$$

where

$$r_1(t) = \prod_{i=1}^{k}\left(t - i - \frac{1}{2}\right), \qquad r_2(t) = \prod_{i=1}^{d-1}(t - d\Delta^i\sqrt{\Delta}).$$

The assumption that $1 \leqslant d < n/55$ ensures that S has $d + k$ distinct elements and furthermore that the elements of S are *interlaced* with the roots of $r_1(t)r_2(t)$, i.e., the open interval between any two consecutive elements of S contains exactly one root of $r_1(t)r_2(t)$. By this interlacing property, for every $t \in S$ the interval (t, ∞) contains precisely $n - t$ roots of $r(t)$. Hence,

$$\operatorname{sgn} r(t) = (-1)^t, \qquad t \in S. \tag{5.6}$$

Continuing,

$$\min_{t \in S}\left|\frac{r(t)}{r(-t)}\right| \geqslant \min_{i=1,\ldots,k+1}\left|\frac{r_1(i)}{r_1(-i)}\right| \cdot \min_{i=1,\ldots,d}\left|\frac{r_2(\lfloor d\Delta^i \rfloor)}{r_2(-\lfloor d\Delta^i \rfloor)}\right| > \exp\left\{-\frac{Cd}{\log(n/d)}\right\}$$

by Lemmas 5.6 and 5.7, where $C > 0$ is an absolute constant. In light of (5.6), we can restate this result as follows:

$$(-1)^t r(t) > \exp\left\{-\frac{Cd}{\log(n/d)}\right\}|r(-t)|, \qquad t \in S.$$

Since r vanishes on $\{-n, \ldots, n\} \setminus (S \cup -S)$ and has degree $\leqslant 2n - 1 - d$, we infer from Theorem 5.2 that $R^+(d, S \cup -S) \geqslant \exp\{-Cd/\log(n/d)\}$. This proves the lower bound for the case $\log n < d < n/55$.

To handle the case $n/55 \leqslant d \leqslant n-1$, a different argument is needed. Let

$$r(t) = (-1)^n\, t \prod_{i=1}^{d} \left(t - i - \frac{1}{2}\right) \prod_{i=d+2}^{n} (t^2 - i^2).$$

By Lemma 5.6, there is an absolute constant $C > 1$ such that

$$\left|\frac{r(t)}{r(-t)}\right| > C^{-d}, \qquad t = 1, 2, \dots, d+1.$$

Since $\operatorname{sgn} r(t) = (-1)^t$ for $t = 1, 2, \dots, d+1$, we conclude that

$$(-1)^t r(t) > C^{-d} |r(-t)|, \qquad t = 1, 2, \dots, d+1.$$

Since r vanishes on $\{-n, \dots, n\} \setminus \{\pm 1, \pm 2, \dots, \pm(d+1)\}$ and has degree $2n-1-d$, we infer from Theorem 5.2 that $R^+(d, \{\pm 1, \pm 2, \dots, \pm(d+1)\}) \geqslant C^{-d}$. This settles the lower bound for the case $n/55 \leqslant d \leqslant n-1$. In fact, this proof works for $\alpha n \leqslant d \leqslant n-1$, where $\alpha > 0$ is any constant.

It remains to prove the upper bound for the case $\log n < d \leqslant n-1$. Here we always have $d \geqslant 2$. Letting $k = \lfloor d/2 \rfloor$ and $\Delta = (n/k)^{1/k}$, define $p \in P_{2k}$ by

$$p(t) = \prod_{i=1}^{k} (t + i) \prod_{i=1}^{k} (t + k\Delta^i).$$

Fix any point $t \in \{1, 2, \dots, n\}$ with $p(-t) \neq 0$. Letting i^* be the integer with $k\Delta^{i^*} < t < k\Delta^{i^*+1}$, we have:

$$\frac{p(t)}{|p(-t)|} > \prod_{i=0}^{i^*} \frac{k\Delta^{i^*+1} + k\Delta^i}{k\Delta^{i^*+1} - k\Delta^i} \prod_{i=i^*+1}^{k} \frac{k\Delta^i + k\Delta^{i^*}}{k\Delta^i - k\Delta^{i^*}} \geqslant \prod_{i=1}^{k} \frac{\Delta^i + 1}{\Delta^i - 1} > \exp\left\{\frac{2(\Delta^k - 1)}{\Delta^k(\Delta - 1)}\right\},$$

where the last inequality holds by Lemma 5.4. Substituting $\Delta = (n/k)^{1/k}$ and recalling that $k \geqslant \Theta(\log n)$, we obtain $p(t) > A|p(-t)|$ for $t = 1, 2, \dots, n$, where

$$A = \exp\left\{\Theta\left(\frac{k}{\log(n/k)}\right)\right\}.$$

As a result, $R^+(2k, \{\pm 1, \pm 2, \dots, \pm n\}) \leqslant 2A/(A^2 + 1)$, the approximant being

$$\frac{A^2 - 1}{A^2 + 1} \cdot \frac{p(t) - p(-t)}{p(t) + p(-t)}. \qquad \square$$

5.3. Equivalence of the majority and sign functions. It remains to prove the equivalence of the majority and sign functions, from the standpoint of approximating them by rational functions on the discrete domain. We have:

THEOREM 5.9. *For every integer d,*

$$R^+(\mathrm{MAJ}_n, d) \leqslant R^+(d-2, \{\pm 1, \pm 2, \dots, \pm \lceil n/2 \rceil\}), \tag{5.7}$$

$$R^+(\mathrm{MAJ}_n, d) \geqslant R^+(d, \{\pm 1, \pm 2, \dots, \pm \lfloor n/2 \rfloor\}). \tag{5.8}$$

Proof. We prove (5.7) first. Fix a degree-$(d-2)$ approximant $p(t)/q(t)$ to $\operatorname{sgn} t$ on $S = \{\pm 1, \dots, \pm\lceil n/2 \rceil\}$, where q is positive on S. For small $\delta > 0$, define

$$A_\delta(t) = \frac{t^2 p(t) - \delta}{t^2 q(t) + \delta}.$$

Then A_δ is a rational function of degree at most d whose denominator is positive on $S \cup \{0\}$. Finally, we have $A_\delta(0) = -1$ and

$$\lim_{\delta \to 0} \max_{t \in S} |\operatorname{sgn} t - A_\delta(t)| = \max_{t \in S} \left| \operatorname{sgn} t - \frac{p(t)}{q(t)} \right|.$$

Then $A_\delta(\frac{1}{2}\sum(x_i+1) - \lfloor n/2 \rfloor)$ is the desired approximant for $\mathrm{MAJ}_n(x_1, \dots, x_n)$.

We now turn to the lower bound, (5.8). For every $\epsilon > R^+(\mathrm{MAJ}_n, d)$, Proposition 2.7 gives a univariate rational function $p(t)/q(t)$ of degree at most d such that for all $x \in \{-1,+1\}^n$, one has

$$\left| \mathrm{MAJ}_n(x) - \frac{p(\sum x_i)}{q(\sum x_i)} \right| < \epsilon$$

and $q(\sum x_i) > 0$. By recalling that $\mathrm{MAJ}_n(x) = \widetilde{\operatorname{sgn}}(\sum x_i)$ and considering the range of $\sum x_i$ separately for n even and n odd, one obtains

$$\max_{t = \pm 1, \pm 2, \dots, \pm\lfloor n/2 \rfloor} \left| \operatorname{sgn} t - \frac{p(2t + (n \bmod 2))}{q(2t + (n \bmod 2))} \right| < \epsilon.$$

This completes the proof of (5.8). ∎

Note that (2.2) and Theorems 5.3, 5.8, and 5.9 immediately imply Theorem 1.7 from the Introduction.

Remark. The above proof of (5.7) illustrates a useful property of univariate rational approximants $A(t) = p(t)/q(t)$ on a finite set S. Specifically, given such an approximant and a point $t^* \notin S$, there exists an approximant A' with $A'(t^*) = a$ for any prescribed value a and $A' \approx A$ everywhere on S. One such construction is

$$A'(t) = \frac{(t - t^*)p(t) + a\delta}{(t - t^*)q(t) + \delta}$$

for sufficiently small $\delta > 0$. Note that A' has degree only 1 higher than the degree of the original approximant, A. This phenomenon is in sharp contrast to approximation by polynomials, which do not possess this corrective ability.

6. Intersections of halfspaces. In this section, we prove our main theorems on the sign-representation of intersections of halfspaces and majority functions. In the two subsections that follow, we give results for the threshold degree as well as *threshold density*, another complexity measure of a sign-representation.

6.1. Lower bounds on the threshold degree. We start by formally stating the theorem of Beigel et al. [10], whose proof was presented in the Introduction and revisited in somewhat greater generality in Remark 3.21.

Theorem 6.1 (Beigel, Reingold, and Spielman). *Let $f\colon X \to \{-1,+1\}$ and $g\colon Y \to \{-1,+1\}$ be given functions, where $X, Y \subset \mathbb{R}^n$ are finite sets. Let d be an integer with $R^+(f,d) + R^+(g,d) < 1$. Then*

$$\deg_\pm(f \wedge g) \leqslant 2d.$$

Recall that Theorem 3.16 gives a converse to Theorem 6.1. We are now in a position to prove our main results on the threshold degree.

THEOREM 6.2 (restatement of Theorems 1.8 and 1.10). *Consider the function* $f\colon \{-1,+1\}^{n^2} \to \{-1,+1\}$ *given by*

$$f(x) = \operatorname{sgn}\left(1 + \sum_{i=1}^{n}\sum_{j=1}^{n} 2^i x_{ij}\right).$$

Let $g\colon \{-1,+1\}^n \to \{-1,+1\}$ *be the majority function on* n *bits. Then*

$$\deg_\pm(f \wedge f) = \Omega(n), \tag{6.1}$$

$$\deg_\pm(\, g \,\wedge\, g\,) = \Omega(\log n). \tag{6.2}$$

Proof. By Theorem 4.8, we have $R^+(f, \epsilon n) \geqslant 1/2$ for some constant $\epsilon > 0$, which settles (6.1) in view of Theorem 3.16. Analogously, Theorems 5.1 and 5.9 show that $R^+(g, \epsilon \log n) \geqslant 1/2$ for some constant $\epsilon > 0$, which settles (6.2) in view of Theorem 3.16. ∎

REMARK. The lower bounds (6.1) and (6.2) are tight [10]. The matching upper bounds can be seen as follows. By Theorem 4.2, we have $R^+(f, Cn) < 1/2$ for some constant $C > 0$, which shows that $\deg_\pm(f \wedge f) = O(n)$ in view of Theorem 6.1. Analogously, Theorems 5.1 and 5.9 imply that $R^+(g, C\log n) < 1/2$ for some constant $C > 0$, which shows that $\deg_\pm(g \wedge g) = O(\log n)$ in view of Theorem 6.1.

We give one additional result, featuring the intersection of the canonical halfspace with a majority function.

THEOREM 1.9 (restated). *Let* $f\colon \{-1,+1\}^{n^2} \to \{-1,+1\}$ *be given by*

$$f(x) = \operatorname{sgn}\left(1 + \sum_{i=1}^{n}\sum_{j=1}^{n} 2^i x_{ij}\right).$$

Let $g\colon \{-1,+1\}^{\lceil\sqrt{n}\rceil} \to \{-1,+1\}$ *be the majority function on* $\lceil\sqrt{n}\rceil$ *bits. Then*

$$\deg_\pm(f \wedge\, g\,) = \Theta(\sqrt{n}).$$

Proof. We prove the lower bound first. Let $\epsilon > 0$ be a suitably small constant. By Theorem 4.8, we have $R^+(f, \epsilon\sqrt{n}) \geqslant 1 - 2^{-\sqrt{n}}$. By Theorems 5.1 and 5.9, we have $R^+(g, \epsilon\sqrt{n}) \geqslant 2^{-\sqrt{n}}$. In view of Theorem 3.16, we conclude that $\deg_\pm(f \wedge g) = \Omega(\sqrt{n})$.

We now turn to the upper bound. Clearly, $R^+(g, \lceil\sqrt{n}\rceil) = 0$ and $R^+(f, 1) < 1$. It follows by Theorem 6.1 that $\deg_\pm(f \wedge g) = O(\sqrt{n})$. ∎

6.2. Lower bounds on the threshold density. In addition to threshold degree, several other complexity measures are of interest when sign-representing Boolean functions by real polynomials. One such is *density,* i.e., the number of distinct monomials in any polynomial that sign-represents a given function. Formally, for a given function $f\colon \{-1,+1\}^n \to \{-1,+1\}$, the *threshold density* $\operatorname{dns}(f)$ is the minimum k such that

$$f(x) \equiv \operatorname{sgn}\left(\sum_{i=1}^{k} \lambda_i \prod_{j \in S_i} x_j\right)$$

for some sets $S_1, \dots, S_k \subseteq \{1, 2, \dots, n\}$ and some reals $\lambda_1, \dots, \lambda_k$. We will show that intersections of two halfspaces not only have high threshold degree but also high threshold density.

We start with the conjunction of two majority functions. Constructions in [10] show that $\mathrm{MAJ}_n \wedge \mathrm{MAJ}_n$ can be sign-represented by a linear combination of $n^{O(\log n)}$ monomials, namely, the monomials of degree up to $O(\log n)$. Klivans and Sherstov [25] complement this with a lower bound of $n^{\Omega(\log n/\log\log n)}$ on the number of distinct monomials needed. Our next result improves this lower bound to a tight $n^{\Theta(\log n)}$.

THEOREM 6.3. *The majority function obeys*

$$\mathrm{dns}(\mathrm{MAJ}_n \wedge \mathrm{MAJ}_n) = n^{\Omega(\log n)}.$$

Proof. Identical to the proof of Klivans and Sherstov [25, Sec. 3.3, Thm. 1.2], with the only difference that Theorem 1.10 should be invoked in place of O'Donnell and Servedio's earlier result [34] that $\deg_\pm(\mathrm{MAJ}_n \wedge \mathrm{MAJ}_n) = \Omega(\log n/\log\log n)$. □

We will now derive an exponential lower bound on the threshold density of the intersection of two halfspaces. For this, we recall an elegant procedure for converting Boolean functions with high threshold degree into Boolean functions with high threshold density, discovered by Krause and Pudlák [27, Prop. 2.1]. Their construction maps a function $f\colon \{-1,+1\}^n \to \{-1,+1\}$ to the function $f^{\mathrm{KP}}\colon (\{-1,+1\}^n)^3 \to \{-1,+1\}$ given by

$$f^{\mathrm{KP}}(x, y, z) = f(\dots, (\overline{z_i} \wedge x_i) \vee (z_i \wedge y_i), \dots).$$

THEOREM 6.4 (Krause and Pudlák). *For every $f\colon \{-1,+1\}^n \to \{-1,+1\}$,*

$$\mathrm{dns}(f^{\mathrm{KP}}) \geqslant 2^{\deg_\pm(f)}.$$

Another ingredient in our analysis is an observation from [25].

LEMMA 6.5 (Klivans and Sherstov). *Let $f\colon \{-1,+1\}^n \to \{-1,+1\}$ be a given Boolean function. Consider any function $F\colon \{-1,+1\}^m \to \{-1,+1\}$ of the form $F(x) = f(\chi_1(x), \dots, \chi_n(x))$, where each χ_i is a (possibly negated) parity function on some subset of the input variables. Then*

$$\mathrm{dns}(f) \geqslant \mathrm{dns}(F).$$

Proof. (Klivans and Sherstov). Immediate from the definition of threshold density and the fact that the product of parity functions is a parity function. □

As a final ingredient of the proof, we record the following observation.

PROPOSITION 6.6. *Fix Boolean functions $f, g\colon \{-1,+1\}^n \to \{-1,+1\}$ and define $f', g'\colon \{-1,+1\}^n \to \{-1,+1\}$ by $f'(x) = -f(-x)$ and $g'(y) = -g(-y)$. Then*

$$\mathrm{dns}(f \wedge g) \geqslant \frac{\mathrm{dns}(f' \wedge g')}{\mathrm{dns}(f)\,\mathrm{dns}(g)},$$

$$\mathrm{dns}(f \wedge g) \geqslant \frac{\mathrm{dns}(f \wedge g')}{\mathrm{dns}(f)}.$$

Proof. Immediate from

$$f'(x) \wedge g'(y) \equiv -(f(-x) \wedge g(-y))f(-x)g(-y),$$
$$f(x) \wedge g'(y) \equiv (f(x) \wedge g(-y))f(x). \qquad \square$$

We are now in a position to prove the desired result for halfspaces.

THEOREM 6.7. *Let* $f_n\colon \{-1,+1\}^{n^2} \to \{-1,+1\}$ *be given by*

$$f_n(x) = \operatorname{sgn}\left(1 + \sum_{i=1}^{n}\sum_{j=1}^{n} 2^i x_{ij}\right).$$

Then

$$\operatorname{dns}(f_n \wedge f_n) = \exp\{\Omega(n)\}, \tag{6.3}$$
$$\operatorname{dns}(f_n \wedge \mathrm{MAJ}_{\lceil\sqrt{n}\rceil}) = \exp\{\Omega(\sqrt{n})\}. \tag{6.4}$$

Proof. Put $m = \lfloor n/4 \rfloor$. The function ${f_m}^{\mathrm{KP}}\colon (\{-1,+1\}^{m^2})^3 \to \{-1,+1\}$ has the representation

$${f_m}^{\mathrm{KP}}(x,y,z) = \operatorname{sgn}\left(1 + \sum_{i=1}^{m}\sum_{j=1}^{m} 2^i (x_{ij} + y_{ij} + x_{ij}z_{ij} - y_{ij}z_{ij})\right).$$

As a result,

$$\begin{aligned}
\operatorname{dns}(f_{4m} \wedge f_{4m}) &\geqslant \operatorname{dns}({f_m}^{\mathrm{KP}} \wedge {f_m}^{\mathrm{KP}}) && \text{by Lemma 6.5}\\
&= \operatorname{dns}((f_m \wedge f_m)^{\mathrm{KP}})\\
&\geqslant 2^{\deg_\pm(f_m \wedge f_m)} && \text{by Theorem 6.4}\\
&\geqslant \exp\{\Omega(m)\} && \text{by Theorem 6.2.}
\end{aligned}$$

By the same argument as in Theorem 4.8, the function f_{4m} is a subfunction of $f_n(x)$ or $-f_n(-x)$. In the former case, (6.3) follows immediately from the lower bound on $\operatorname{dns}(f_{4m} \wedge f_{4m})$. In the latter case, (6.3) follows from the lower bound on $\operatorname{dns}(f_{4m} \wedge f_{4m})$ and Proposition 6.6.

The proof of (6.4) is entirely analogous, with Theorem 1.9 invoked instead of Theorem 6.2. $\square$

Krause and Pudlák's method generalizes to linear combinations of conjunctions rather than parity functions. In other words, if a function $f\colon \{-1,+1\}^n \to \{-1,+1\}$ has threshold degree d and

$$f^{\mathrm{KP}}(x,y,z) \equiv \operatorname{sgn}\left(\sum_{i=1}^{N} \lambda_i T_i(x,y,z)\right)$$

for some conjunctions $T_1, T_2, \ldots, T_N$ of literals from among $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, $\neg x_1, \neg y_1, \neg z_1, \ldots, \neg x_n, \neg y_n, \neg z_n$, then $N \geqslant 2^{\Omega(d)}$. With this in mind, Theorems 6.3 and 6.7 and their proofs adapt easily to this alternate definition of density.

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