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Efficient Inference of Partial Types

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Abstract

Partial types for the λ-calculus were introduced by Thatte in 1988 [8] as a means of typing objects that are not typable with simple types, such as heterogeneous lists and persistent data. In that paper he showed that type inference for partial types was semidecidable. Decidability remained open until quite recently, when O'Keefe and Wand [5] gave an exponential time algorithm for type inference.

In this paper we give an $O(n^3)$ algorithm. Our algorithm constructs a certain finite automaton that represents a canonical solution to a given set of type constraints. Moreover, the construction works equally well for recursive types; this solves an open problem stated in [5].

1 Introduction

Partial types for the pure λ-calculus were introduced by Thatte in 1988 [8] as a way to type certain λ-terms that are untypable in the simply-typed λ-calculus. They are of substantial pragmatic value, since they allow the

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typing of such constructs as heterogeneous lists and persistent data that would otherwise be untypable.

Formally, partial types comprise a partially ordered set $(T, \leq)$, where T is the set of well-formed terms over the constant symbol Ω and the binary type constructor $\rightarrow$, and $\leq$ is the partial order defined inductively as follows:

(i) $t \leq \Omega$ for any t;

(ii) $s \rightarrow t \leq s' \rightarrow t'$ if and only if $s' \leq s$ and $t \leq t'$.

Intuitively, the type constructor $\rightarrow$ represents the usual function space constructor, and Ω is a *universal type* that includes every other type. The partial order $\leq$ can be thought of as *type inclusion* or *coercion*; that is, $s \leq t$ if it is possible to coerce type s into type t.

Clause (ii) in the definition of $\leq$ models the fact that a function with domain s and range t can be coerced to a function with domain s' and range t' provided s' can be coerced to s and t can be coerced to t'; thus the coercion order on functions is covariant in the range and contravariant in the domain. That $\leq$ is contravariant in the domain is considered to be the main source of difficulty in type inference algorithms.

If E is a λ-term, t is a partial type, and A is a type environment, *i.e.* a partial function assigning types to variables, then the judgement

$$A \ \vdash \ E : t \tag{1}$$

means that E has the partial type t in the environment A. Formally, this holds when the judgement (1) is derivable using the following four rules, first proposed by Thatte [8]:

$$A \vdash x : t \quad (\text{provided } A(x) = t) \tag{2}$$

$$\frac{A[x \leftarrow s] \vdash E : t}{A \vdash \lambda x.E : s \rightarrow t} \tag{3}$$

$$\frac{A \vdash E : s \rightarrow t \quad A \vdash F : s}{A \vdash EF : t} \tag{4}$$

$$\frac{A \vdash E : s \quad s \leq t}{A \vdash E : t} \tag{5}$$

The first three rules are the usual rules for simple types and the last rule is the rule of *subsumption*.

More λ-terms are typable with partial types than with simple types. For example, the term $\lambda f.(fK(fI))$, where $K = \lambda x.(\lambda y.x)$ and $I = \lambda z.z$, has partial type

$$(\Omega \to (\Omega \to \Omega)) \quad \to \quad \Omega \ ,$$

but no simple type.

As with any type discipline, the question of *type inference* is of paramount importance:

> Given a λ-term E, is E typable? If so, give a type for it.

For this particular discipline, the type inference question can be reduced to the problem of solving a finite system of *type constraints*, which are just inequalities over terms with type variables. Rephrased, the problem becomes:

> Given a system of inequalities of the form $s \leq t$, where s and t are terms over $\to$ and variables ranging over T, does the system have a solution in $(T, \leq)$? If so, give a solution.

We give this reduction and prove its correctness in Section 2.

In his original paper [8], Thatte showed that the type inference problem for partial types is semidecidable. The problem of decidability remained unsolved until quite recently, when O'Keefe and Wand [5] presented an exponential time algorithm. Their algorithm involves iterated substitution and gives no hint of the possibility of the existence of canonical solutions; indeed, there exist satisfiable constraint systems with no $\leq$-minimal solution.

In this paper we show that the type inference problem for partial types is solvable in time $O(n^3)$, where n is the size of the λ-term. Moreover, the solutions we construct are *canonical* in the sense that they are least in the so-called *Böhm order*, a natural order different from $\leq$.

Our algorithm constructs a certain finite automaton with $O(n^2)$ states from the given system of type constraints. The canonical solution to the system is just the regular language accepted by the automaton, where we represent types as binary trees and binary trees as prefix-closed sets of strings over a two-letter alphabet. In this representation, the Böhm order is just set inclusion $\subseteq$.

The canonical solution always exists, but it may not be finite; however, since it is contained in all other solutions, we can check for the existence of a

finite solution by checking whether the canonical solution is finite. Thus the typability question reduces essentially to the finiteness problem for regular sets.

Our construction works equally well for recursive types; this solves an open problem stated in [5].

Henglein [2] has shown that the type inference problem for partial types is P-hard; thus it is P-complete. It can also be shown that every λ-term with a partial type is strongly normalizing [9] and that every λ-term in normal form has a partial type [6].

Despite the fact that our polynomial-time algorithm now makes automatic type inference for partial types feasible, we feel that the more important contribution of this work is theoretical: namely, the precise mathematical characterization of the set of solutions to a system of type constraints and the identification of a canonical solution. We hope that the automata-theoretic approach developed here will be useful in dealing with other type systems.

2 From Rules to Constraints

In this section we rephrase the type inference problem in terms of solutions of finite systems of type constraints. This isolates the essential combinatorial structure of the problem independent of type-theoretic syntax.

Given a λ-term E, the type inference question can be rephrased in terms of solving a system of type constraints. Assume that E has been α-converted so that all bound variables are distinct. Let X be the set of λ-variables x occurring in E, and let Y be a set of variables disjoint from X consisting of one variable $[\![F]\!]$ for each occurrence of a subterm F of E. (The notation $[\![F]\!]$ is ambiguous because there may be more than one occurrence of F in E. However, it will always be clear from context which occurrence is meant.) We generate the following system of inequalities over $X \cup Y$:

- for every occurrence in E of a subterm of the form $\lambda x.F$, the inequality

$$x \to [\![F]\!] \;\leq\; [\![\lambda x.F]\!] \;;$$

- for every occurrence in E of a subterm of the form FG, the inequality

$$[\![F]\!] \;\leq\; [\![G]\!] \to [\![FG]\!] \;;$$

- for every occurrence in E of a λ-variable x, the inequality

$$x \leq [\![x]\!] .$$

Denote by $C(E)$ the system of constraints generated from E in this fashion. We show below that the solutions of $C(E)$ over T correspond to the possible type annotations of E in a sense made precise by Theorem 2.1.

Let A be a type environment assigning a type to each λ-variable occurring freely in E. If L is a function assigning a type to each variable in $X \cup Y$, we say that L *extends* A if A and L agree on the domain of A.

Theorem 2.1 *The judgement $A \vdash E : t$ is derivable if and only if there exists a solution L of $C(E)$ extending A such that $L([\![E]\!]) = t$. In particular, if E is closed, then E is typable with type t if and only if there exists a solution L of $C(E)$ such that $L([\![E]\!]) = t$.*

Proof. We first prove that if $C(E)$ has such a solution L, then $L \vdash E : L([\![E]\!])$ is derivable. We proceed by induction on the structure of E.

For the base case, $L \vdash x : L([\![x]\!])$ is derivable using rules (2) and (5), since $L(x) \leq L([\![x]\!])$.

For the induction step, consider first $\lambda x.E$. To derive $L \vdash \lambda x.E : L([\![\lambda x.E]\!])$, by rule (5) and the fact that $L(x) \to L([\![E]\!]) \leq L([\![\lambda x.E]\!])$ it suffices to derive $L \vdash \lambda x.E : L(x) \to L([\![E]\!])$. By rule (3) it suffices to derive $L[x \leftarrow L(x)] \vdash E : L([\![E]\!])$, or in other words $L \vdash E : L([\![E]\!])$. But since L is a solution of $C(\lambda x.E)$, it is also a solution of $C(E)$, thus the desired derivation is provided by the induction hypothesis.

Now consider EF. Since L is a solution of $C(EF)$, it is also a solution of $C(E)$ and $C(F)$. From the induction hypothesis, we obtain derivations of $L \vdash E : L([\![E]\!])$ and $L \vdash F : L([\![F]\!])$. Moreover, since L is a solution of $C(EF)$, $L([\![E]\!]) \leq L([\![F]\!]) \to L([\![EF]\!])$. Then $L([\![E]\!])$ must be of the form $s \to t$ where $L([\![F]\!]) \leq s$ and $t \leq L([\![EF]\!])$. Using rule (5), we can derive $L \vdash F : s$. Using rule (4), we can derive $L \vdash EF : t$. Using rule (5) again, we can derive $L \vdash EF : L([\![EF]\!])$.

Conversely, suppose $A \vdash E : t$ is derivable, and consider a derivation of minimal length. Since the derivation is minimal, there is exactly one application of the rule (4) involving a particular occurrence of a subterm FG, exactly one application of the rule (3) involving the subterm $\lambda x.F$, and

exactly one application of the rule (2) involving a particular occurrence of a λ-variable x. In the last case, there is a unique type s such that $B(x) = s$ for any B such that a judgement $B \vdash G : u$ appears in the derivation for some occurrence of a subterm G of $\lambda x.F$; this can be proved by induction on the structure of the derivation of $B \vdash G : u$. Finally, there can be at most one application of the rule (5) involving a particular occurrence of any subterm; if there were more than one, they could be combined using the transitivity of $\leq$ to give a shorter derivation.

Now construct L as follows. For every λ-variable x occurring freely in E, define $L(x) = A(x)$. For every bound λ-variable x, let $\lambda x.F$ be the subterm of E in which it is bound, find the last judgement in the derivation of the form $B \vdash F : s$ involving that occurrence of F, and define $L(x) = B(x)$. Finally, for every occurrence of a subterm F of E, find the last judgement in the derivation of the form $B \vdash F : s$ involving that occurrence of F, and define $L([\![F]\!]) = s$.

Certainly L extends A and $L([\![E]\!]) = t$. We now show that L is a solution of $C(E)$.

For an occurrence of a subterm of the form $\lambda x.F$, find the unique application of the rule (3) deriving the judgement $B \vdash \lambda x.F : s \to u$ from the premise $B[x \leftarrow s] \vdash F : u$. Then $L(x) = s$, $L([\![F]\!]) = u$, and $L(x) \to L([\![F]\!]) = s \to u \leq L([\![\lambda x.F]\!])$.

For an occurrence of a subterm of the form EF, find the unique application of the rule (4) deriving the judgement $B \vdash EF : u$ from the premises $B \vdash E : s \to u$ and $B \vdash F : s$. Then $L([\![E]\!]) = s \to u$, $L([\![F]\!]) = s$, and $u \leq L([\![EF]\!])$, thus $L([\![E]\!]) \leq L([\![F]\!]) \to L([\![EF]\!])$.

Finally, for an occurrence of a bound λ-variable x, find the unique application of the rule (3) deriving the judgement $B \vdash \lambda x.F : s \to u$ from the premise $B[x \leftarrow s] \vdash F : u$. Then $L(x) = s$. The rule (2) must have been applied to obtain a judgement of the form $B' \vdash x : L(x)$ and only rule (5) applied to that occurrence of x thereafter, thus $L(x) \leq L([\![x]\!])$. □

A similar constraint system was used without proof in [7]. Except for a minor misstatement in the formulation of [7], the two constraint systems are equivalent.

3 From Types to Trees

Partial types are essentially binary trees, which can be represented as as certain sets of strings over the binary alphabet $\{L, R\}$. In this section we develop some elementary properties of this representation and generalize to infinite trees.

Definition 3.1 Let $\alpha, \beta, \ldots$ denote elements of $\{L, R\}^*$. The *parity* of α is the number mod 2 of L's in α. The parity of α is denoted $\pi\alpha$. A string α is said to be *even* (respectively, *odd*) if $\pi\alpha = 0$ (respectively, 1).

A *tree* is a subset $\sigma \subseteq \{L, R\}^*$ that is

- nonempty,
- closed under prefix, and
- binary, in the sense that for all α, $\alpha R \in \sigma$ iff $\alpha L \in \sigma$.

We use $\sigma, \tau, \ldots$ to denote trees. The set of all trees is denoted $\widehat{T}$.

A tree is *finite* if it is finite as a set of strings. A *path* in a tree σ is a maximal subset of σ linearly ordered by the prefix relation. By König's Lemma, a tree is finite iff it has no infinite paths.

An element $\alpha \in \sigma$ is a *leaf* of σ if it is not a proper prefix of any other element of σ. □

For $A, B \subseteq \{L, R\}^*$ and $\alpha \in \{L, R\}^*$, define

$$\begin{aligned} A \cdot B &= \{\epsilon\} \cup \{L\alpha \mid \alpha \in A\} \cup \{R\beta \mid \beta \in B\} \\ A{\downarrow}\alpha &= \{\beta \mid \alpha\beta \in A\} \ . \end{aligned}$$

For trees σ, τ, $\sigma \cdot \tau$ is the tree with left subtree σ and right subtree τ, and $\sigma{\downarrow}\alpha$ is the subtree of σ at α if $\alpha \in \sigma$, $\emptyset$ if not.

The following lemma establishes some elementary properties of the operators $\cdot$ and $\downarrow$ on trees.

Lemma 3.2

(i) $(\sigma \cdot \tau){\downarrow}L = \sigma$ *and* $(\sigma \cdot \tau){\downarrow}R = \tau$

(ii) $\sigma = \sigma{\downarrow}L \cdot \sigma{\downarrow}R$

(iii) $(\sigma \downarrow \alpha) \downarrow \beta = \sigma \downarrow \alpha\beta$

(iv) α *is a leaf of* σ *iff* $\sigma \downarrow \alpha = \{\epsilon\}$.

Proof. All properties are immediate consequences of the definitions. □

The types T are in a natural one-to-one correspondence with the finite trees in $\widehat{T}$ under the embedding $e : T \to \widehat{T}$ given by

$$\begin{aligned} e(\Omega) &= \{\epsilon\} \\ e(s \to t) &= e(s) \cdot e(t) \ . \end{aligned}$$

Under this embedding, an occurrence of Ω at the fringe of a type s corresponds to a leaf of $e(s)$.

We now wish to define a partial order on $\widehat{T}$ that agrees with the order $\leq$ on T under the embedding e.

Definition 3.3 For $\sigma, \tau \in \widehat{T}$, define $\sigma \leq \tau$ if both of the following conditions hold for any α:

(i) if α is an even leaf of σ, then $\alpha R \notin \tau$;

(ii) if α is an odd leaf of τ, then $\alpha R \notin \sigma$.

□

Lemma 3.4 *The relation* $\leq$ *is a partial order on trees, and agrees with the order* $\leq$ *on types under the embedding* e*. In particular, for any* σ, τ, σ_i, τ_i,

(i) $\sigma \leq \{\epsilon\}$;

(ii) $\{\epsilon\} \leq \tau$ *if and only if* $\tau = \{\epsilon\}$;

(iii) $\sigma_1 \cdot \sigma_2 \leq \tau_1 \cdot \tau_2$ *if and only if* $\tau_1 \leq \sigma_1$ *and* $\sigma_2 \leq \tau_2$.

Proof. We first show that $\leq$ is a partial order. It is trivially reflexive. To show transitivity, let $\sigma \leq \tau \leq \omega$ and assume for a contradiction that α is an even leaf of σ and $\alpha R \in \omega$. Let β be the longest prefix of αR in τ. If $\beta = \alpha R$, then Definition 3.3(i) is violated for σ, τ. Otherwise β is a prefix of α and a leaf of τ. If β is even, then Definition 3.3(i) is violated for τ, ω. If β is odd,

then Definition 3.3(ii) is violated for σ, τ. In all three cases we contradict the assumption $\sigma \leq \tau \leq \omega$. A symmetric argument under the assumption that α is an odd leaf of ω and $\alpha R \in \sigma$ likewise leads to a contradiction.

For antisymmetry, assume $\sigma \leq \tau \leq \sigma$. Let $\alpha \in \sigma$ and let β be the longest prefix of α in τ. If $\beta \neq \alpha$, then β is a leaf of τ; but then either β is even, which contradicts $\tau \leq \sigma$, or β is odd, which contradicts $\sigma \leq \tau$. Thus $\beta = \alpha$ and $\alpha \in \tau$. Since $\alpha \in \sigma$ was arbitrary, $\sigma \subseteq \tau$. A symmetric argument shows that $\tau \subseteq \sigma$.

We next establish the properties (i)–(iii) in turn.

(i) If α is an even leaf in σ, then clearly $\alpha R \notin \{\epsilon\}$. There are no odd leaves in $\{\epsilon\}$.

(ii) The *if* follows by reflexivity; *only if* follows by (i) and antisymmetry.

(iii) Let $\sigma = \sigma_1 \cdot \sigma_2$ and $\tau = \tau_1 \cdot \tau_2$. For *if*, assume that α is an even leaf in σ. We proceed by induction on the length of α. The case $\alpha = \epsilon$ is not possible. If $\alpha = L\beta$ then β is an odd leaf in σ_1, so $\beta R \notin \tau_1$, so $L\beta R \notin \tau_1 \cdot \tau_2$, so $\alpha R \notin \tau$. If $\alpha = R\beta$ then β is an even leaf in σ_2, so $\beta R \notin \tau_2$, so $R\beta R \notin \tau_1 \cdot \tau_1$, so $\alpha R \notin \tau$. Assume now that α is an odd leaf in τ. We proceed by induction on the length of α. The case $\alpha = \epsilon$ is not possible. If $\alpha = L\beta$ then β is an even leaf in τ_1, so $\beta R \notin \sigma_1$, so $L\beta R \notin \sigma_1 \cdot \sigma_2$, so $\alpha R \notin \sigma$. If $\alpha = R\beta$ then β is an odd leaf in τ_2, so $\beta R \notin \sigma_2$, so $R\beta R \notin \sigma_1 \cdot \sigma_2$, so $\alpha R \notin \sigma$.

For *only if*, assume that α is an even leaf in τ_1; then $L\alpha$ is an odd leaf in τ, so $L\alpha R \notin \sigma$, so $\alpha R \notin \sigma_1$. If α is an odd leaf in σ_1, then $L\alpha$ is an even leaf in σ, so $L\alpha R \notin \tau$, so $\alpha R \notin \tau_1$. If α is an even leaf in σ_2, then $R\alpha$ is an even leaf in σ, so $R\alpha R \notin \tau$, so $\alpha R \notin \tau_2$. If α is an odd leaf in τ_2, then $R\alpha$ is an odd leaf in τ, so $R\alpha R \notin \sigma$, so $\alpha R \notin \sigma_2$.

Finally, we show that the order on types agrees with the order on trees under the embedding e, *i.e.*, $s \leq t$ if and only if $e(s) \leq e(t)$. We proceed by induction on the structure of s and t. If $t = \Omega$ then the result follows from (i). If $s = \Omega$ then the result is immediate from (ii). If $s = s_1 \to s_2$ and $t = t_1 \to t_2$ then the induction hypothesis tells us that $t_1 \leq s_1$ if and only if $e(t_1) \leq e(s_1)$ and $s_2 \leq t_2$ if and only if $e(s_2) \leq e(t_2)$. The result now follows from (iii) and the definitions of e and $\leq$ on types. □

Amadio and Cardelli [1] give an alternative definition of a partial order on recursive types involving infinite chains of finite approximations. Definition 3.3 is equivalent to theirs [4].

Lemma 3.5 *The following properties hold for all σ, τ.*

(i) $(R \in \sigma \ \wedge \ R \in \tau \ \wedge \ \sigma \leq \tau) \ \Rightarrow \ (\sigma \downarrow L \geq \tau \downarrow L) \ \wedge \ (\sigma \downarrow R \leq \tau \downarrow R)$;

(ii) $(\sigma \leq \tau \ \wedge \ R \in \tau) \ \Rightarrow \ R \in \sigma$.

Proof. Property (i) follows immediately from Lemma 3.4(iii), and (ii) follows immediately from Lemma 3.4(ii). □

4 From Constraints to Graphs

Instead of systems of type constraints involving type variables, we consider a more general notion of a *constraint graph*.

Definition 4.1 A *constraint graph* is a directed graph $G = (S, L, R, \leq)$ consisting of a set of nodes S and three sets of directed edges $L, R, \leq$. We write $s \xrightarrow{L} t$ to indicate that the pair (s, t) is in the edge set L, and similarly $s \xrightarrow{R} t$, $s \xrightarrow{\leq} t$. A constraint graph must satisfy the properties:

- any node has at most one outgoing L edge and at most one outgoing R edge;
- a node has an outgoing L edge if and only if it has an outgoing R edge.

A *solution* for G is any map $h : S \to \widehat{T}$ such that

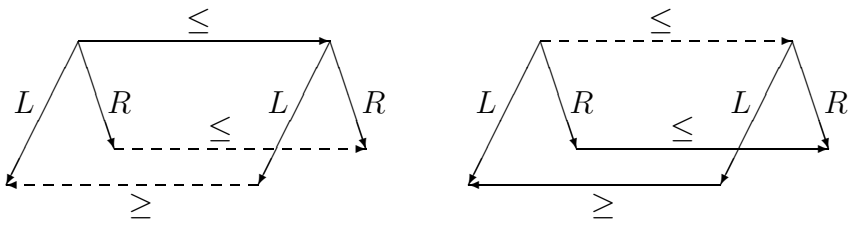
(i) if $u \xrightarrow{L} v$ and $u \xrightarrow{R} w$, then $h(u) = h(v) \cdot h(w)$;

(ii) if $u \xrightarrow{\leq} v$, then $h(u) \leq h(v)$.

The solution h is *finite* if $h(s)$ is a finite set for all s. □

A system of type constraints as described in §2 gives rise to a constraint graph by associating a unique node with every subexpression occurring in the system of constraints, defining L and R edges from an occurrence of an expression to its left and right subexpressions, and defining $\leq$ edges for the inequalities.

Definition 4.2 A constraint graph is *closed* if the edge relation $\leq$ is reflexive, transitive, and closed under the following two rules which say that the dashed edges exist whenever the solid ones do:



Note that these two rules resemble the two implications of Lemma 3.4(iii). The *closure* of a constraint graph G is the smallest closed graph containing G as a subgraph. □

Lemma 4.3 *A constraint graph and its closure have the same set of solutions.*

Proof. Any solution of the closure of G is also a solution of G, since G has fewer constraints. Conversely, the closure of G can be constructed from G by iterating the closure rules, and it follows inductively by Lemma 3.4 that any solution of G satisfies the additional constraints added by this process. □

5 From Graphs to Automata

In this section we define two automata $\mathcal{M}$ and $\mathcal{N}$ and describe their relationship. These automata will be used to characterize the canonical solution of a given constraint graph G. An intuitive account follows the formal definitions.

Definition 5.1 Let a closed constraint graph $G = (S, L, R, \leq)$ be given. The automaton $\mathcal{M}$ is defined as follows. The input alphabet of $\mathcal{M}$ is $\{L, R\}$. The states of $\mathcal{M}$ are $S^2 \cup S^1 \cup S^0$. States in S^2 are written (s, t), those in S^1 are written (s), and the unique state in S^0 is written $(\ )$. The transitions

are defined as follows.

$$
\begin{array}{ll}
(u,v) \xrightarrow{\epsilon} (u,v') & \text{if } v \stackrel{\leq}{\rightarrow} v' \text{ in } G \\
(u,v) \xrightarrow{\epsilon} (u',v) & \text{if } u' \stackrel{\leq}{\rightarrow} u \text{ in } G \\
(u,v) \xrightarrow{R} (u',v') & \text{if } u \xrightarrow{R} u' \text{ and } v \xrightarrow{R} v' \text{ in } G \\
(u,v) \xrightarrow{L} (v',u') & \text{if } u \xrightarrow{L} u' \text{ and } v \xrightarrow{L} v' \text{ in } G \\
(u,v) \xrightarrow{\epsilon} (v) & \text{always} \\
(v) \xrightarrow{\epsilon} (v') & \text{if } v \stackrel{\leq}{\rightarrow} v' \text{ in } G \\
(v) \xrightarrow{R} (v') & \text{if } v \xrightarrow{R} v' \text{ in } G \\
(v) \xrightarrow{L} (\ ) & \text{if } v \xrightarrow{L} v' \text{ in } G
\end{array}
$$

If p and q are states of $\mathcal{M}$ and $\alpha \in \{L,R\}^*$, we write $p \xrightarrow{\alpha} q$ if the automaton can move from state p to state q under input α, including possible ϵ-transitions.

The automaton $\mathcal{M}_s$ is the automaton $\mathcal{M}$ with start state (s,s). All states are accept states; thus the language accepted by $\mathcal{M}_s$ is the set of strings α for which there exists a state p such that $(s,s) \xrightarrow{\alpha} p$. We denote this language by $\mathcal{L}(s)$. □

Informally, we can think of the automaton $\mathcal{M}_s$ as follows. We start with two pebbles, one green and one red, on the node s of the constraint graph G. We can move the green pebble forward along a $\leq$ edge at any time, and we can move the red pebble backward along a $\leq$ edge at any time. We can move both pebbles simultaneously along R edges leading out of the nodes they occupy. We can also move them simultaneously along outgoing L edges, but in the latter case we switch their colors. At any time, we can elect to remove the red pebble; thereafter, we can move the green pebble forward along $\leq$ or R edges as often as we like, and forward along an L edge once, at which point the pebble must be removed. The sequence of L's and R's that were seen gives a string in $\mathcal{L}(s)$, and all strings in $\mathcal{L}(s)$ are obtained in this way.

The intuition motivating the definition of $\mathcal{M}$ is that we want to identify the conditions that require a path to exist in any solution. Thus $\mathcal{L}(s)$ is the set of α that *must* be there; this intuition is made manifest in Lemma 5.2. It turns out that once we identify this set, we are able to show that it is a solution itself.

We now show that $\mathcal{M}$ accepts only essential strings.

Lemma 5.2 *If $h : S \to \widehat{T}$ is any solution and $(s,s) \xrightarrow{\alpha} p$, then $\alpha \in h(s)$. Moreover,*

(i) if $p = (u,v)$ then $h(u) \leq h(s) \downarrow \alpha \leq h(v)$;

(ii) if $p = (v)$ then $h(s) \downarrow \alpha \leq h(v)$.

Proof. We proceed by induction on the number of transitions. If this is zero, then $p = (s,s)$ and $\alpha = \epsilon$, and the result is immediate. Otherwise, assume that $(s,s) \xrightarrow{\alpha} p$ and the lemma holds for this sequence of transitions. We argue by cases, depending on the form of the next transition out of p.

If p if of the form (u,v), then the induction hypothesis says that $\alpha \in h(s)$ and $h(u) \leq h(s) \downarrow \alpha \leq h(v)$.

If $(u,v) \xrightarrow{\epsilon} (u',v')$, then $u' \xrightarrow{\leq} u$ and $v \xrightarrow{\leq} v'$, so $\alpha\epsilon = \alpha \in h(s)$ and

$$h(u') \;\leq\; h(u) \;\leq\; h(s) \downarrow \alpha \;\leq\; h(v) \;\leq\; h(v') \;.$$

If $(u,v) \xrightarrow{R} (u',v')$, then $u \xrightarrow{R} u'$ and $v \xrightarrow{R} v'$, so $h(u') = h(u) \downarrow R$ and $h(v') = h(v) \downarrow R$. Then $R \in h(v)$, so by Lemmas 3.2 and 3.5, $R \in h(s) \downarrow \alpha$ and $\alpha R \in h(s)$, and

$$h(u') \;=\; h(u) \downarrow R \;\leq\; h(s) \downarrow \alpha R \;\leq\; h(v) \downarrow R \;=\; h(v') \;.$$

If $(u,v) \xrightarrow{L} (v',u')$, then $u \xrightarrow{L} u'$ and $v \xrightarrow{L} v'$, so $h(u') = h(u) \downarrow L$ and $h(v') = h(v) \downarrow L$. Then $L \in h(v)$, so by Lemmas 3.2 and 3.5, $L \in h(s) \downarrow \alpha$ and $\alpha L \in h(s)$, and

$$h(u') \;=\; h(u) \downarrow L \;\geq\; h(s) \downarrow \alpha L \;\geq\; h(v) \downarrow L \;=\; h(v') \;.$$

If $(u,v) \xrightarrow{\epsilon} (v)$ then $\alpha\epsilon = \alpha \in h(s)$ and $h(s) \downarrow \alpha \leq h(v)$.

If p if of the form (v), then the induction hypothesis says that $\alpha \in h(s)$ and $h(s) \downarrow \alpha \leq h(v)$.

If $(v) \xrightarrow{\epsilon} (v')$, then $v \xrightarrow{\leq} v'$, so $\alpha\epsilon = \alpha \in h(s)$ and

$$h(s) \downarrow \alpha \;\leq\; h(v) \;\leq\; h(v') \;.$$

If $(v) \xrightarrow{R} (v')$, then $v \xrightarrow{R} v'$, so $h(v') = h(v) \downarrow R$. Then $R \in h(v)$, so by Lemmas 3.2 and 3.5, $R \in h(s) \downarrow \alpha$ and $\alpha R \in h(s)$, and

$$h(s) \downarrow \alpha R \;\leq\; h(v) \downarrow R \;=\; h(v') \;.$$

Finally, if $(v) \xrightarrow{L} (\;)$, then $v \xrightarrow{L} v'$, so $h(v') = h(v) \downarrow L$. Then $L \in h(v)$, so by Lemmas 3.2 and 3.5, $L \in h(s) \downarrow \alpha$ and $\alpha L \in h(s)$. □

Here we give a useful alternative characterization of $\mathcal{L}(s)$ in terms of a different automaton $\mathcal{N}$.

Definition 5.3 Let $G = (S, L, R, \leq)$ be given as above. We define the automaton $\mathcal{N}$ over the input alphabet $\{L, R\}$ as follows. The states of $\mathcal{N}$ are $S \times \{0, 1\}$; we use square brackets for states of $\mathcal{N}$ to distinguish them from states of $\mathcal{M}$. The transitions are

$$\begin{array}{ll} [s,0] \xrightarrow{\epsilon} [t,0] & \text{if } s \overset{\leq}{\Rightarrow} t \text{ in } G \\ [s,1] \xrightarrow{\epsilon} [t,1] & \text{if } t \overset{\leq}{\Rightarrow} s \text{ in } G \\ [s,b] \xrightarrow{R} [t,b] & \text{if } s \xrightarrow{R} t \text{ in } G \\ [s,b] \xrightarrow{L} [t,\overline{b}] & \text{if } s \xrightarrow{L} t \text{ in } G. \end{array}$$

As above, we write $[s,b] \xrightarrow{\alpha} [t,c]$ if $[s,b]$ can go to $[t,c]$ under α, including possible ϵ-transitions. □

The automaton $\mathcal{N}$ has states $[s,b]$ where b is a Boolean value. The second component is used to keep track of the parity of the scanned string. We can think of $[s,b]$ as a pebble on s; the second component gives the color of the pebble. If the pebble is green ($b = 0$), we can move it forward along $\leq$ edges. If the pebble is red ($b = 1$), we can move it backward along $\leq$ edges. We can move the pebble forward along R or L edges at any time, but if we move it along an L edge, then we switch the color.

The following lemmas relate $\mathcal{M}$ and $\mathcal{N}$.

Lemma 5.4

(i) $(s,s) \xrightarrow{\alpha} (u,v)$ *if and only if both*

- *(a)* $[s, \pi\alpha] \xrightarrow{\alpha} [v, 0]$
- *(b)* $[s, \overline{\pi\alpha}] \xrightarrow{\alpha} [u, 1]$.

(ii) $(s) \xrightarrow{\alpha} (t)$ *if and only if* $\alpha = R^k$ *for some* k *and* $[s,0] \xrightarrow{\alpha} [t,0]$.

(iii) $(s) \xrightarrow{\alpha} (\ )$ *if and only if* $\alpha = R^k L$ *for some* k *and* $[s,0] \xrightarrow{\alpha} [t,1]$ *for some* t.

Proof. We prove the three parts in turn. In each case we proceed by induction on α.

(i) If $\alpha = \epsilon$ then

$$\begin{aligned}(s,s) \xrightarrow{\epsilon} (u,v) &\Leftrightarrow u \overset{\leq}{\Rightarrow} s \ \wedge \ s \overset{\leq}{\Rightarrow} v \\ &\Leftrightarrow [s,0] \xrightarrow{\epsilon} [v,0] \ \wedge \ [s,1] \xrightarrow{\epsilon} [u,1] \\ &\Leftrightarrow [s,\pi\epsilon] \xrightarrow{\epsilon} [v,0] \ \wedge \ [s,\overline{\pi\epsilon}] \xrightarrow{\epsilon} [u,1] \ .\end{aligned}$$

If $\alpha = \beta L$ then

$$(s,s) \xrightarrow{\beta} (p,q) \xrightarrow{L} (q',p') \xrightarrow{\epsilon} (u,v) \ .$$

By the induction hypothesis, this is equivalent to

$$\begin{aligned}&[s,\pi\beta] \xrightarrow{\beta} [q,0] \ \wedge \ [s,\overline{\pi\beta}] \xrightarrow{\beta} [p,1] \wedge \\ &\quad p \xrightarrow{L} p' \ \wedge \ q \xrightarrow{L} q' \ \wedge \ u \overset{\leq}{\Rightarrow} q' \ \wedge \ p' \overset{\leq}{\Rightarrow} v \\ &\Leftrightarrow \ [s,\pi\beta] \xrightarrow{\beta} [q,0] \xrightarrow{L} [q',1] \xrightarrow{\epsilon} [u,1] \ \wedge \\ &\qquad\quad [s,\overline{\pi\beta}] \xrightarrow{\beta} [p,1] \xrightarrow{L} [p',0] \xrightarrow{\epsilon} [v,0] \\ &\Leftrightarrow \ [s,\overline{\pi\beta}] \xrightarrow{\beta L} [v,0] \ \wedge \ [s,\pi\beta] \xrightarrow{\beta L} [u,1] \\ &\Leftrightarrow \ [s,\pi\beta L] \xrightarrow{\beta L} [v,0] \ \wedge \ [s,\overline{\pi\beta L}] \xrightarrow{\beta L} [u,1] \ .\end{aligned}$$

If $\alpha = \beta R$ then

$$(s,s) \xrightarrow{\beta} (p,q) \xrightarrow{R} (p',q') \xrightarrow{\epsilon} (u,v) \ .$$

By the induction hypothesis, this is equivalent to

$$\begin{aligned}&[s,\pi\beta] \xrightarrow{\beta} [q,0] \ \wedge \ [s,\overline{\pi\beta}] \xrightarrow{\beta} [p,1] \wedge \\ &\quad p \xrightarrow{R} p' \ \wedge \ q \xrightarrow{R} q' \ \wedge \ u \overset{\leq}{\Rightarrow} p' \ \wedge \ q' \overset{\leq}{\Rightarrow} v \\ &\Leftrightarrow \ [s,\pi\beta] \xrightarrow{\beta} [q,0] \xrightarrow{R} [q',0] \xrightarrow{\epsilon} [v,0] \ \wedge \\ &\qquad\quad [s,\overline{\pi\beta}] \xrightarrow{\beta} [p,1] \xrightarrow{R} [p',1] \xrightarrow{\epsilon} [u,1] \\ &\Leftrightarrow \ [s,\pi\beta] \xrightarrow{\beta R} [v,0] \ \wedge \ [s,\overline{\pi\beta}] \xrightarrow{\beta R} [u,1] \\ &\Leftrightarrow \ [s,\pi\beta R] \xrightarrow{\beta R} [v,0] \ \wedge \ [s,\overline{\pi\beta R}] \xrightarrow{\beta R} [u,1] \ .\end{aligned}$$

(ii) If $\alpha = \epsilon$ then

$$\begin{aligned}(s) \xrightarrow{\epsilon} (t) &\Leftrightarrow s \overset{\leq}{\Rightarrow} t \\ &\Leftrightarrow \epsilon = R^0 \ \wedge \ [s,0] \xrightarrow{\epsilon} [t,0] \ .\end{aligned}$$

If $\alpha = \beta R$ then

$$(s) \xrightarrow{\beta} (p) \xrightarrow{R} (p') \xrightarrow{\epsilon} (t) \ .$$

By the induction hypothesis, this is equivalent to

$$\begin{aligned} &\beta = R^k \ \wedge \ [s,0] \xrightarrow{\beta} [p,0] \ \wedge \ p \xrightarrow{R} p' \ \wedge \ p' \xrightarrow{\leq} t \\ \Leftrightarrow \ \ &\beta R = R^{k+1} \ \wedge \ [s,0] \xrightarrow{\beta R} [t,0] \ . \end{aligned}$$

The case $\alpha = \beta L$ is not possible.

(iii) The cases $\alpha = \epsilon$ and $\alpha = \beta R$ are not possible. If $\alpha = \beta L$ then

$$(s) \xrightarrow{\beta} (p) \xrightarrow{L} (\ ) \ .$$

Using (ii), this is equivalent to

$$\begin{aligned} &\beta = R^k \ \wedge \ [s,0] \xrightarrow{\beta} [p,0] \ \wedge \ p \xrightarrow{L} t \\ \Leftrightarrow \ \ &\alpha = R^k L \ \wedge \ [s,0] \xrightarrow{\beta L} [t,1] \ . \end{aligned}$$

□

Lemma 5.5 *For any string α, $\alpha \in \mathcal{L}(s)$ if and only if there exist β, k, u, v such that*

(i) $\alpha = \beta R^k$ *or* $\alpha = \beta R^k L$,

(ii) $[s, \pi\beta] \xrightarrow{\alpha} [v, \pi\alpha \oplus \pi\beta]$, *and*

(iii) $[s, \overline{\pi\beta}] \xrightarrow{\beta} [u, 1]$.

Here $\oplus$ denotes addition mod 2.

Proof. First assume $\alpha \in \mathcal{L}(s)$. Then $(s,s) \xrightarrow{\alpha} p$ for some state p. If $p = (u,v)$ then $(s,s) \xrightarrow{\alpha} (u,v)$, so by Lemma 5.4 we have

$$[s, \pi\alpha] \xrightarrow{\alpha} [v, 0] \ \wedge \ [s, \overline{\pi\alpha}] \xrightarrow{\alpha} [u, 1] \ .$$

But then we can choose $\alpha = \beta$ and $k = 0$. If $p = (v)$ then for some β, γ we have

$$(s,s) \xrightarrow{\beta} (u,q) \xrightarrow{\epsilon} (q) \xrightarrow{\gamma} (v)$$

so by Lemma 5.4 we have

$$\gamma = R^k \ \wedge \ [s, \pi\beta] \xrightarrow{\beta} [q, 0] \xrightarrow{\gamma} [v, 0] \ \wedge \ [s, \overline{\pi\beta}] \xrightarrow{\beta} [u, 1] \ .$$

Since $\pi\alpha = \pi\beta$, this is equivalent to

$$\alpha = \beta R^k \ \wedge \ [s, \pi\beta] \xrightarrow{\alpha} [v, \pi\alpha \oplus \pi\beta] \ \wedge \ [s, \overline{\pi\beta}] \xrightarrow{\beta} [u, 1]$$

and we are done. If $p = (\ )$ then for some β, γ we have

$$(s, s) \xrightarrow{\beta} (u, q) \xrightarrow{\epsilon} (q) \xrightarrow{\gamma} (\ )$$

so by Lemma 5.4 we have

$$\gamma = R^k L \ \wedge \ [s, \pi\beta] \xrightarrow{\beta} [q, 0] \xrightarrow{\gamma} [v, 1] \ \wedge \ [s, \overline{\pi\beta}] \xrightarrow{\beta} [u, 1] \ .$$

Since $\pi\alpha \neq \pi\beta$, this is equivalent to

$$\alpha = \beta R^k L \ \wedge \ [s, \pi\beta] \xrightarrow{\alpha} [v, \pi\alpha \oplus \pi\beta] \ \wedge \ [s, \overline{\pi\beta}] \xrightarrow{\beta} [u, 1]$$

and we are done.

Conversely, assume (i)–(iii). We have $\alpha = \beta\gamma$ where $\gamma = R^k$ or $R^k L$, and

$$[s, \pi\beta] \xrightarrow{\alpha} [v, \pi\gamma]$$
$$[s, \overline{\pi\beta}] \xrightarrow{\beta} [u, 1] \ .$$

We must have

$$[s, \pi\beta] \xrightarrow{\beta} [p, 0] \xrightarrow{\gamma} [v, \pi\gamma]$$

for some p. From Lemma 5.4 we have

$$(s, s) \xrightarrow{\beta} (u, p) \xrightarrow{\epsilon} (p)$$

and either $(p) \xrightarrow{\gamma} (v)$ or $(p) \xrightarrow{\gamma} (\ )$, depending on whether $\gamma = R^k$ or $R^k L$. In either case $\alpha \in \mathcal{L}(s)$. $\square$

6 Main Result

In this section we prove the main result: that $\mathcal{L}(s)$ gives the canonical solution of G.

Theorem 6.1 *The sets $\mathcal{L}(s)$ are trees, and the function $\mathcal{L} : S \to \widehat{T}$ is a solution of G. Moreover, if $h : S \to \widehat{T}$ is any other solution, then $\mathcal{L}(s) \subseteq h(s)$ for any s.*

Proof. We first show that $\mathcal{L}(s) \in \widehat{T}$. It is clearly nonempty, since $(s, s) \xrightarrow{\epsilon} (s, s)$; it is prefix closed by definition; and it is binary because G always has L and R edges in pairs.

In order to show that $\mathcal{L}$ is a solution of G, we need to show

(i) if $u \xrightarrow{L} v$ and $u \xrightarrow{R} w$, then $\mathcal{L}(u) = \mathcal{L}(v) \cdot \mathcal{L}(w)$;

(ii) if $u \xrightarrow{\leq} v$, then $\mathcal{L}(u) \leq \mathcal{L}(v)$.

First, we show (i) as two inclusions. Assume that $\alpha \in \mathcal{L}(v) \cdot \mathcal{L}(w)$. We proceed by induction on α. If $\alpha = \epsilon$ then we are done, since $\epsilon \in \mathcal{L}(u)$. If $\alpha = L\beta$ then $\beta \in \mathcal{L}(v)$, so $(v, v) \xrightarrow{\beta} p$ for some p. From

$$(u, u) \xrightarrow{L} (v, v) \xrightarrow{\beta} p$$

we conclude that $L\beta \in \mathcal{L}(u)$. If $\alpha = R\beta$ then $\beta \in \mathcal{L}(w)$, so $(w, w) \xrightarrow{\beta} p$ for some p. From

$$(u, u) \xrightarrow{R} (w, w) \xrightarrow{\beta} p$$

we conclude that $R\beta \in \mathcal{L}(u)$.

Assume that $\alpha \in \mathcal{L}(u)$. We proceed by induction on α. If $\alpha = \epsilon$ then we are done, since $\epsilon \in \mathcal{L}(v) \cdot \mathcal{L}(w)$. If $\alpha = L\beta$ then from Lemma 5.5 there exist γ, k, p, and q such that $\beta = \gamma R^k$ or $\beta = \gamma R^k L$ and

$$[u, \pi L\gamma] \xrightarrow{\alpha} [p, \pi\alpha \oplus \pi L\gamma] \ \wedge \ [u, \overline{\pi L\gamma}] \xrightarrow{L\gamma} [q, 1]$$

Since $[u, \pi L\gamma] \xrightarrow{L} [v, \pi\gamma]$, $[u, \overline{\pi L\gamma}] \xrightarrow{L} [v, \overline{\pi\gamma}]$, and $\pi\alpha \oplus \pi L\gamma = \pi\beta \oplus \pi\gamma$ it follows that

$$[v, \pi\gamma] \xrightarrow{\beta} [p, \pi\beta \oplus \pi\gamma] \ \wedge \ [v, \overline{\pi\gamma}] \xrightarrow{\gamma} [q, 1]$$

so $\beta \in \mathcal{L}(v)$ and $\alpha \in \mathcal{L}(v) \cdot \mathcal{L}(w)$. If $\alpha = R\beta$ then from Lemma 5.5 there exist γ, k, p, and q such that $\beta = \gamma R^k$ or $\beta = \gamma R^k L$ and

$$[u, \pi R\gamma] \xrightarrow{\alpha} [p, \pi\alpha \oplus \pi R\gamma] \ \wedge \ [u, \overline{\pi R\gamma}] \xrightarrow{R\gamma} [q, 1]$$

Since $[u, \pi R\gamma] \xrightarrow{R} [w, \pi\gamma]$, $[u, \overline{\pi R\gamma}] \xrightarrow{R} [w, \overline{\pi\gamma}]$, and $\pi\alpha \oplus \pi R\gamma = \pi\beta \oplus \pi\gamma$ it follows that

$$[w, \pi\gamma] \xrightarrow{\beta} [p, \pi\beta \oplus \pi\gamma] \ \wedge \ [w, \overline{\pi\gamma}] \xrightarrow{\gamma} [q, 1]$$

so $\beta \in \mathcal{L}(w)$ and $\alpha \in \mathcal{L}(v) \cdot \mathcal{L}(w)$.

Second, we show (ii). We need to show that for any u, v, α,

- if $u \xrightarrow{\leq} v$, α even, $\alpha \in \mathcal{L}(u)$, and $\alpha R \in \mathcal{L}(v)$, then $\alpha R \in \mathcal{L}(u)$;
- if $u \xrightarrow{\leq} v$, α odd, $\alpha \in \mathcal{L}(v)$, and $\alpha R \in \mathcal{L}(u)$, then $\alpha R \in \mathcal{L}(v)$.

Using the characterization in terms of $\mathcal{N}$, these two cases can be rolled into one: it suffices to show for any s, t, α,

- if $[s, \pi\alpha] \xrightarrow{\epsilon} [t, \pi\alpha]$, $\alpha \in \mathcal{L}(s)$, and $\alpha R \in \mathcal{L}(t)$, then $\alpha R \in \mathcal{L}(s)$.

By Lemma 5.5, we have

$$[s, \pi\beta] \xrightarrow{\alpha} [u, \pi\alpha \oplus \pi\beta] \tag{6}$$
$$[s, \overline{\pi\beta}] \xrightarrow{\beta} [v, 1] \tag{7}$$
$$[t, \pi\beta'] \xrightarrow{\alpha R} [u', \pi\alpha R \oplus \pi\beta'] \tag{8}$$
$$[t, \overline{\pi\beta'}] \xrightarrow{\beta'} [v', 1] \ .$$

Since αR ends with R, we must have $\alpha R = \beta' R^k$ for some k and $\pi\alpha R = \pi\beta'$; thus from (8) and $[s, \pi\alpha] \xrightarrow{\epsilon} [t, \pi\alpha]$ we get

$$[s, \pi\alpha] \xrightarrow{\alpha R} [u', 0] \ . \tag{9}$$

If $\pi\alpha = \pi\beta$, we use (9) and (7) to get

$$[s, \pi\beta] \xrightarrow{\alpha R} [u', \pi\alpha \oplus \pi\beta]$$
$$[s, \overline{\pi\beta}] \xrightarrow{\beta} [v, 1] \ .$$

If $\pi\alpha \neq \pi\beta$, we use (9) and (6) to get

$$[s, \pi\alpha] \stackrel{\alpha R}{\rightarrow} [u', \pi\alpha R \oplus \pi\alpha]$$
$$[s, \overline{\pi\alpha}] \stackrel{\alpha}{\rightarrow} [u, 1] \ .$$

In either case we have $\alpha R \in \mathcal{L}(s)$ by Lemma 5.5.

To show that $\mathcal{L}$ is minimal, we need to show that for any other solution $h : S \rightarrow \widehat{T}$, $\mathcal{L}(s) \subseteq h(s)$ for all s. This follows directly from Lemma 5.2. $\Box$

Recursive types are just regular trees [1]. The canonical solution we have constructed, although possibly infinite, is a regular tree. Thus we have solved the type inference problem for recursive types left open in [5]. Specifically, given a λ-term, we construct the corresponding constraint graph and automaton $\mathcal{M}$. Every subterm corresponds to a node s in the constraint graph, and its Böhm-minimal type annotation is represented by the language $\mathcal{L}(s)$.

Note that the Böhm-minimal type of any typable λ-term trivially is Ω. What we compute is the unique minimal Church-style explicit type annotation. For simple types we have that types and type annotations are isomorphic. This is not so for partial types. For example, the Böhm-minimal type of $\lambda f.(fK(fI))$ is just Ω, but its Böhm-minimal type annotation, which our algorithm computes, is

$$\lambda f : (\Omega \rightarrow (\Omega \rightarrow \Omega)).(f(\lambda x : \Omega.\lambda y : \Omega.x)(f(\lambda z : \Omega.z)))$$

In the following section we give an efficient decision procedure for the existence of a *finite* type.

7 An Algorithm

We have argued that the type inference problem studied in [8, 5] is equivalent to the following: given a finite constraint graph G, does G have a finite solution? Using the characterization of the previous section, we can answer this question easily.

Theorem 7.1 *One can decide in time $O(n^3)$ whether a constraint graph of size n has a finite solution.*

Proof. By Theorem 6.1, there exists a finite solution if and only if the canonical solution is finite. To determine this, we need only check whether any $\mathcal{L}(s)$ contains an infinite path. We first form the constraint graph, then close it; this gives a graph with n vertices and $O(n^2)$ edges. This can be done in time $O(n^3)$. We then form the automaton $\mathcal{M}$, which has n^2+n+1 states but only $O(n^3)$ transitions, at most $O(n)$ from each state. We then check for a cycle with at least one non-ϵ transition reachable from some (s,s). This can be done in linear time in the size of the graph using depth-first search. The entire algorithm requires time $O(n^3)$. □

A λ-term of size n yields a constraint graph with $O(n)$ nodes and $O(n)$ edges. Allowing types to be represented succinctly by the automata $\mathcal{M}_s$, we have

Corollary 7.2 *The type inference problem for partial types with or without recursive types is solvable in time $O(n^3)$.*

8 A Characterization of All Solutions

We have shown that any solution $h : S \to \widehat{T}$ of a constraint graph $G = (S, L, R, \leq)$ contains the canonical solution $\mathcal{L}$ in the sense that $\mathcal{L}(s) \subseteq h(s)$ for all $s \in S$. However, there certainly exist functions $h : S \to \widehat{T}$ containing $\mathcal{L}$ in this sense that are not solutions. In this section we give a precise characterization of the set of all solutions of G.

In the following, we write $\mathcal{L}_G$ for $\mathcal{L}$ to denote the dependence on G.

Theorem 8.1 *Let $G = (S, L, R, \leq)$ be a constraint graph. A function $h : S \to \widehat{T}$ is a solution of G if and only if there exists a (possibly infinite) constraint graph $G' = (S', L', R', \leq')$ containing G as a subgraph such that $h = \mathcal{L}_{G'}$ on S.*

Proof. First we show that if G is a subgraph of G', then $\mathcal{L}_{G'}$ gives a solution of G. Suppose $u, v, w \in S$, $u \xrightarrow{L} v$, and $u \xrightarrow{R} w$. Since G is a subgraph of G', $u \xrightarrow{L'} v$ and $u \xrightarrow{R'} w$. By Theorem 6.1, $\mathcal{L}_{G'}$ is a solution of G', therefore $\mathcal{L}_{G'}(u) = \mathcal{L}_{G'}(v) \cdot \mathcal{L}_{G'}(w)$. Similarly, if $u, v \in S$ and $u \xrightarrow{\leq} v$, then $u \xrightarrow{\leq'} v$, therefore $\mathcal{L}_{G'}(u) \leq \mathcal{L}_{G'}(v)$. The two conditions of Definition 4.1 are met.

Conversely, let $h : S \to \widehat{T}$ be any solution of G. We construct a constraint graph G' from h containing G as a subgraph and show that h and $\mathcal{L}_{G'}$ agree on S. Define

$$\begin{aligned} S' &= S \cup \{(s,\alpha) \mid s \in S,\ \alpha \in h(s)\} \\ L' &= L \cup \{((s,\alpha),(s,\alpha L)) \mid s \in S,\ \alpha L \in h(s)\} \\ R' &= R \cup \{((s,\alpha),(s,\alpha R)) \mid s \in S,\ \alpha R \in h(s)\} \\ \leq' &= \leq \cup \{((s,\epsilon),s) \mid s \in S\} \cup \{(s,(s,\epsilon)) \mid s \in S\}\ . \end{aligned}$$

The graph $G' = (S', L', R', \leq')$ is a constraint graph, since each node has an L' successor iff it has an R' successor, and L' and R' successors are unique.

We show now that h agrees with $\mathcal{L}_{G'}$ on S. If $\alpha \in h(s)$, then $(s,\alpha) \in S'$, and there is a path from (s,ϵ) to (s,α) in G' with label α. In the automaton $\mathcal{M}'$ constructed from G', $(s,s) \xrightarrow{\alpha} ((s,\alpha),(s,\alpha))$, thus $\alpha \in \mathcal{L}_{G'}(s)$.

To show the reverse inclusion, we extend h in a natural way to a solution h' of G', and then appeal to Theorem 6.1 to conclude that h' contains $\mathcal{L}_{G'}$. Define

$$\begin{aligned} h'(s) &= h(s) \\ h'(s,\alpha) &= h(s) \downarrow \alpha\ . \end{aligned}$$

It remains to show that h' is a solution of G'. If $s,t,u \in S$, $s \xrightarrow{L} t$, and $s \xrightarrow{R} u$, then

$$h'(s) \;=\; h(s) \;=\; h(t) \cdot h(u) \;=\; h'(t) \cdot h'(u)\ .$$

If αL, $\alpha R \in h(s)$, then by Lemma 3.2,

$$\begin{aligned} h'(s,\alpha) &= h(s) \downarrow \alpha \\ &= (h(s) \downarrow \alpha) \downarrow L \cdot (h(s) \downarrow \alpha) \downarrow R \\ &= h(s) \downarrow \alpha L \cdot h(s) \downarrow \alpha R \\ &= h'(s,\alpha L) \cdot h'(s,\alpha R)\ . \end{aligned}$$

Finally, if $s,t \in S$ and $s \xrightarrow{\leq} t$, then

$$h'(s) \;=\; h(s) \;\leq\; h(t) \;=\; h'(t)\ ,$$

and to satisfy the inequalities $s \leq' (s,\epsilon) \leq' s$ we have

$$h(s) \;=\; h(s) \downarrow \epsilon \;=\; h'(s,\epsilon)\ .$$

□

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References

[1] Roberto M. Amadio and Luca Cardelli. Subtyping recursive types. In *Proc. 18th Symp. Princip. Programming Lang.*, pages 104–118. ACM, January 1991.

[2] Fritz Henglein. Personal communication, 1992.

[3] Dexter Kozen, Jens Palsberg, and Michael I. Schwartzbach. Efficient inference of partial types. In *Proc. 33rd Symp. Found. Comput. Sci.*, pages 363–371. IEEE, October 1992.

[4] Dexter Kozen, Jens Palsberg, and Michael I. Schwartzbach. Efficient recursive subtyping. In *Proc. 20th ACM Symp. Princip. Programming Lang.*, pages 419–428. ACM, January 1993.

[5] Patrick M. O'Keefe and Mitchell Wand. Type inference for partial types is decidable. In *Proc. ESOP'92, European Symposium on Programming*, volume 582 of *Lect. Notes in Comput. Sci.*, pages 408–417. Springer-Verlag, 1992.

[6] Jens Palsberg. Normal forms have partial types. *Information Processing Letters*, 45(1):1–3, January 1993.

[7] Jens Palsberg and Michael I. Schwartzbach. Safety analysis versus type inference for partial types. *Infor. Processing Letters*, 43(4):175–180, September 1992.

[8] Satish Thatte. Type inference with partial types. In *Proc. International Colloquium on Automata, Languages, and Programming 1988*, volume 317 of *Lect. Notes in Comput. Sci.*, pages 615–629. Springer-Verlag, 1988.

[9] Mitchell Wand and Patrick M. O'Keefe. Partially typed terms are strongly normalizing. Manuscript, December 1991.