Patterns in Numbers

Allen Klinger

February 8, 2006

Allen Klinger, Professor
Computer Science Department
University of California Los Angeles
Los Angeles, California 90095-1596
klinger@cs.ucla.edu

Keywords Consecutive Squares, Square Pyramid, Sum Identity, Sum of Squares Identity, Pythagorean, Sums

Abstract

Key identities and fundamental notions of mathematics such as symmetry appear in an exposition linking three relationships - groups of equations - involving consecutive integers.

1 Introduction

Mathematics structures pattern ideas. Two authors [1], [5] point to *patterns* leading to *mathematics* as a scientific (organized) body of knowledge. *Number patterns* described here are regularities and related occurrences that came to the attention of ordinary people long before there was an organized or formal field called mathematics. Several situations involving quantity-patterning described here illuminate useful identities that are basic to the field.

Observing the regularities we call *number patterns*, is akin to laboratory study. Education via a laboratory approach is also called doing experiments. Many in education see that as a way to stimulate and enliven learning the knowledge in a field. Experiments are like number patterns: both reveal questions that led to the current body of knowledge. Familiar (and not-so-well-known) formulae result from experience in observing patterns. This will be shown through identities and pattern relations below.

Pattern awareness may be the human skill that led to a much of the knowledge we study in school. Whether that is so or not, this paper presents three numerical issues together, to convey how mathematics grew from pattern observation.

2 Background

A mathematical puzzle in a magazine column [2] included this key statement:

> We are interested in finding a sequence of 2n + 1 consecutive positive integers, such that the sum of the squares of the first n + 1 integers equals the sum of the squares of the last n integers. The simplest such sequence is
>
> $$3^2 + 4^2 = 5^2 \tag{1}$$

While students often learn about the *Pythagorean theorem* and some specific instances, neither the name nor the values in any such triple possess power to stimulate. I found my own imagination jolted by the second equation (next section) expressing the idea of this statement, and hope others will be similarly impacted.

Sources for this article include [6], an unavailable letter [7] mentioned there, and [8]. The latter presents the first four such sequences. Using an observation by [7], [6] presents the first five. However none of the available sources call attention to pattern thinking. When it is used, this pattern case [*i.e.,* (1)-(3); see equations beginning the next section] and a simpler example [equations (4)-(6) below], quickly lead to mathematical concepts.

3 Observations

What value begins the series of adjacent squares? Gardner wrote that Linton [7] gave a formula for the lowest value in an equation like the two that follow:

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2 \tag{2}$$

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2 \tag{3}$$

Instead of the lowest value, view this through pattern thinking. That leads first to looking at these three equations in terms of symmetry, and then to the general case. Start with the number of entries on the left side: always one more than how many are on the right. Symmetry is present (and is the most useful general pattern notion). The symmetry is about the largest of the numbers (before squaring) on the left (expression with most entries).

Divide the numbers that are squared into three groups: those lower than the value closest to, and left of, the equals sign; that number itself; and those on the right side of the equality.

In (1)-(3), but actually any consecutive squares equation, all terms are derived by counting from the particular value we just isolated (one of three groups). That value is always immediately left of the equals sign when there is one more term at its left than on the right. If we call the isolated value (with location just left of the equals sign when the larger number of consecutive squares is on the left side) *the pivot,* numbers to be squared on the left are obtained by counting down from it (those at right, up).

There are other expressions not involving squares that are very similar to the above equations, for example (4)-(6). In each case (1)-(6) there is a single number from which all others in that equation are found by counting down or counting up. Equation (7) states this algebraically for the (4)-(6) case.

$$1 + 2 = 3 \tag{4}$$

$$4 + 5 + 6 = 7 + 8 \tag{5}$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15 \tag{6}$$

$$(t - a) + (t - a + 1) + ... + t = (t + 1) + ... + (t + a) \tag{7}$$

We see just a-many entries on the RHS of (7). By collecting all terms with t's on the LHS and all without t's on the RHS we obtain the equality of items on the extremes of (8), while the value t clearly comes from subtracting the terms where it appears multiplied by a and *(a + 1)*:

$$(a + 1)t - a(t) = t = 2(sum - of - numbers - from - 1 - to - a) \tag{8}$$

Writing the equation (8) parenthetical expression in standard mathematical notation gives:

$$t/2 = \sum_{i=1}^{a} i = a(a + 1)/2 \tag{9}$$

The relationship at the right side of equation (9) is readily demonstrated by writing two rows of numbers 1 through a, one below the other but in reverse order. Vertically adding similarly-located values gets just a entries that all are *(a + 1)* in value. But that addition yields *sum-of-the-numbers-from-1-to-a* twice:

$$2\sum_{i=1}^{a} i = a(a+1) \tag{10}$$

We now know that in an expression like (4)-(6) a number just left of the equals sign (side with larger number of terms) has value t ; that t begins a counting down and up process that is behind such an equality; and that t has value calculated as follows using only the number of terms on the right (fewer entries), a :

$$t = a(a+1) \tag{11}$$

Linton's formula for that lowest value, which we'll call b, in terms of a for the number of entries on the right side of the equation, is:

$$b = a(2a+1) \tag{12}$$

4 Squares

The significant issue raised by [2] is, for consecutive squares, not what lowest value appears in such expressions [*e.g.,* (1)-(3)], but *how is the pivot to be found* . Begin with expressing the consecutive square property by extending (7):

$$(t-a)^2 + (t-a+1)^2 + ... + t^2 = (t+1)^2 + ... + (t+a)^2 \tag{13}$$

Again, the terms - numbers that are squared - are in three groups. One consists solely of *the pivot,* t . All other numbers to be squared are obtained by counting down or up from it. But we know:

$$(c-d)^2 = c^2 - 2cd + d^2 = (c^2 + 2cd + d^2) - 4cd = (c+d)^2 - 4cd \tag{14}$$

The property displayed in (14) applies to (13) and creates a simple result. When two numbers are added or subtracted and the result is squared, they yield the same squared terms and only differ in the sign of their cross-products. Every *minus* element in (13) has a corresponding *plus* term except t^2 itself. This means that (13) can be rewritten:

$$t^2 = 4\sum_{i=1}^{a} ti \tag{15}$$

This simplifies to:

$$t = 4\sum_{i=1}^{a} i \tag{16}$$

But due to (10), the expression for sums of integers from *1* to *n,* (16) can be restated more simply in an expression similar to equation (11) for the linear case, as:

$$t = 4\sum_{i=1}^{a} i = 2a(a+1) \tag{17}$$

5 Values

To see how the identities may have come out of observing patterns, we coin words for the above concepts, where the algebra used symbols. *E.g.,* in the analysis a was helpful. Let's call that the *order.* The symbol b can stand for *base* or *bottom.* That value is each consecutive square equation's *lowest value.* The observed values for order and base of equations (1)-(3) follow in tabular form (Table 1). Notice that the product of *order* and *number of entries* is the *base* (*entries* refers to the total number of terms or elements in an equation). Stated algebraically, this is Linton's formula, (12), namely $a(2a + 1) = b$.

order	*base*	*number of entries*
1	3	3
2	10	5
3	21	7

Table 1: Product Pattern

Another table, Table 2, shows symmetry about the pivot. It also displays pattern relationships between characteristics of different order equations.

(count	down	from)	*pivot*	=	(count	up	from)
		3	*4*	=	5		
	10	11	*12*	=	13	14	
21	22	23	*24*	=	25	26	27

Table 2: Symmetry Pattern

Note that the number of left side terms times the largest value on the right divided by the number of right side terms is the base of the next higher order equation. Because we are counting up from the pivot the largest value on the right is $t + a$. Hence this statement can be rendered algebraically as:

$$(t_a + a_a)(a_a + 1)/a_a = b_{a+1} \tag{18}$$

Since (12) holds in general, (18) can be rewritten using $b_{a+1} = (a+1)(2a+3)$:

$$(t+a)(a+1)/a = (a+1)(2a+3) \tag{19}$$

Algebraic manipulation quickly yields the extremes of (17), $t = 2a(a+1)$.

6 Square Sum of Summed Squares

A series of consecutive squares summed begins with unity. As the number of elements summed increases, an interesting fact appears. The total is only square when last number squared is twenty-four [4] . (For more detail on this question see [3] ; proof is beyond the scope of this item.)

Watson's proof depends on another summing squares relationship, one that equates the sum of squares beginning at one and progressing through the integers to some value n as the product of three factors in n. The relationship is:

$$1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6 \tag{20}$$

Although 12, 18, 24, 30 and 36 all are evenly divided by 6, the only one of these candidates for n so that (17) is square is $n = 24$.

The way to establish (17) begins with the first few values of such a sum as n increases from unity. One can find these as 1, 5, 14, 30, and 55 by direct computation and then notice each satisfies the right hand side of equation (17). With (17) known true at some value, it can be proven by a simple process. The process involves this observation: if it holds at say n *implies or causes* it to also be valid at *(n+1)*, then as long as it is true for some case n it must be true for any n. Its truth at the five numerical values indicates a pattern. The patterns makes it worth investigating this relationship:

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6} \tag{21}$$

Evaluating the two expressions in (18) by multiplying out terms gives the same result. Hence (17) is true in general.

Watson demonstrated $n = 24$ is the only value so a sum of numbers squared from *1* to n itself is square (*viz.,(17).*) $n = 24$ causes the sum value from (17) to be $4900 = 70^2$.

7 Conclusion

Mathematics began with agriculture and commerce. We could describe it as *numbers, geometry and trigonometry set down with proof and rigor.* We have seen three pattern relationships involving numbers and two identities that concisely portray notions present in them.

Today's version of this mathematical field of knowledge grew. It is now a complex structure that many of those at the learning, and some at the teaching, side of education, find intimidating. The complexity is in part the result of how the early knowledge has been influenced and expanded by needs in economics, medicine, physics, engineering, and computing. Historically proof and relevance to new knowledge (*e.g., irrational numbers* in mathematics), made (1) and the Pythagorean theorem central, and (2) unimportant. But what is most important is to keep interest alive in both students and their teachers. The number relationships displayed here can be used to address this issue. There is great value in maintaining a sense of wonder about mathematical issues and elements. The above exposition examined numbers, patterns, identities, and proof-like reasoning. We hope that these cases offer students and teachers a way to stimulate mathematical learning.

8 Acknowledgement

Taylor B. Kang and Michelle M. Amagrande of California State Polytechnic University, Pomona, observed the relationships described in **Values**. Richard Korf, University of California Los Angeles (UCLA), told me of the problem Watson solved. David G. Cantor, UCLA and Center for Communications Research, San Diego, suggested documenting **Squares**.

References

[1] Steen, Lynn Arthur . The Science of Patterns. *Science*, pages 611–616, April 1988.

[2] Brain ticklers. The Bent of Tau Beta Pi, Spring 2004, p. 44; Available on the world-wide web as http://www.tbp.org/pages/publications/BTs/Sp04.pdf, 2004.

[3] E. W. Weisstein. Square Pyramidal Number. MathWorld; http://mathworld.wolfram.com/SquarePyramidalNumber.html, 1999.

[4] G. N. Watson. The Problem of the Square Pyramid. *Messenger. Math. 48*, pages 1–22, 1918.

[5] Keith J. Devlin. *Mathematics: The Science of Patterns.* W. H. Freeman and Co., NY, 1994.

[6] Martin Gardner. *The Numerology of Dr. Matrix.* Simon and Schuster, NY, 1967.

[7] Russell Linton. Letter to Martin Gardner, 1967.

[8] T. H. Beldon. Runs of Squares. *The Mathematical Gazette*, pages 334–335, December 1961.